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STRIPPING A MATRIX OF ITS ZERO EIGENVALUES

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Besides being an algebraist,
I firmly believe Gauß elimination
is a better place to learn mathematics than
Kronecker-style epsilon-⋆⋆⋆⋆ing...

Adrien Deloro

Abstract

Of course everyone knows it, but the Gauß elimination is spectral theory for $\lambda = 0$.

Preface

I teach linear algebra to first year undergraduates, and one of my more repetitive duties as a teacher involves convincing students that row operations do not preserve the eigenvalues of a matrix.

So I was alarmed to discover that a slight enhancement of the Gauss elimination procedure allows us to produce, from a degenerate square matrix A, a smaller, and invertible, matrix B which has the same Jordan cells as the Jordan cells of A for its non-zero eigenvalues. In particular, eigenvalues of B are precisely non-zero eigenvalues (with their multiplicities) of the matrix A.

1. *Algorithm*

The disheartening realisation described in the Preface came to me when two facts collided in my brain.

Reduction to echelon form. Let A be a $n \times n$ matrix of rank r, with $r < n$ and P its echelon form; then it is well-known

$$A = JP,$$

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where J is an invertible matrix and P has form

$$P = \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where R is $r \times n$ matrix made of the first r (non-zero) rows of P. Let Q be the $n \times r$ matrix made from the first r columns of J; obviously,

$$A = QR.$$

This decomposition is very useful for proving properties of rank of matrix [†].

Comparing Jordan cells of $A = QR$ and $B = RQ$. Now I use a line from [‡]: it is easy to check that

$$\begin{bmatrix} QR & 0 \\ R & 0 \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ R & RQ \end{bmatrix}.$$

Therefore the matrices

$$\begin{bmatrix} QR & 0 \\ R & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ R & RQ \end{bmatrix}$$

are conjugate and therefore QR and RQ have the same Jordan cells for non-zero eigenvalues.

Iteration. So, given a matrix A with eigenvalue 0, we can construct a smaller matrix $B = RQ$ which has the same Jordan cells for non-zero eigenvalues as the matrix A has. Iterating that, we eventually come to a non-degenerate matrix B. As simple as that.

Iterations may be needed: try this procedure on the matrix A which is already in echelon form and decomposed into Jordan cells:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

After the first iteration you get

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and after the second the 1×1 matrix $[1]$, as expected.

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†W. P. Wardlaw, Row rank equals column rank. Mathematics Magazine 78 no. 4 (2005), 316–318.

‡C. R. Johnson and E. A. Schreiner, The relationship between AB and BA. The American Mathematical Monthly, 103 no. 7 (Aug.-Sep. 1996), 578–582.

2. *Discussion*

But Didactylos posed the famous philosophical conundrum:
*"Yes, But What"s It **Really** All About, Then,*
When You Get Right Down To It,
*I Mean **Really**?"*

Terry Pratchett

But what is the geometric meaning of all that?

After closer look, you find the underlying – and demystified – form of the algorithm for stripping zeroes.

Here it is.

Let $V = \mathbb{R}^n$.

– Compute the eigenspace E for the eigenvalue 0, that is, the space of solutions of the homogeneous system of simultaneous equations $A\mathbf{v} = \mathbf{0}$.
  This, of course, amounts to decomposition $A = JP$ as in my original algorithm.
– Calculate an image of A in the factor space V/E.
  This means computing the conjugate of A:

$$J^{-1}AJ = J^{-1} \cdot JP \cdot J = PJ.$$

  This matrix is block-diagonal; let us denote by B the $r \times r$ block in the left upper corner; obviously, B is an image of A in V/E and therefore has the same Jordan cells for non-zero eigenvalues as the matrix A.
– Iterate until the matrix has no zero eigenvalues.

Obviously, this is exactly my original algorithm.

As simple as that.

3. *Generalisation*

Of course, this procedure can be modified to strip a matrix A of other eigenvalues: it λ is the one you want to get rid off, strip $A - \lambda I$ of its eigenvalues 0, then replace the resulting matrix B by $B + \lambda I$, where the identity matrices I are of matching dimensions.

4. *The moral of this fable*

The Gauß elimination is spectral theory for $\lambda = 0$.

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About the Author

I am a research mathematician but I have 40+ years of teaching experience at secondary school and university level in four different countries with four different education systems; since 1998 I am a Professor of Pure Mathematics at the University of Manchester.

I also have an interest in cognitive aspects of mathematical practice; see my book *Mathematics under the Microscope*, [‡], which explains a mathematician's outlook at psycho-physiological and cognitive issues in mathematics and mathematics education. Some of my papers on mathematics education can be found in my personal online journal/blog *Selected Passages From Correspondence With Friends* [§].

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[‡] A. V. Borovik, *Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice.* Amer. Math. Soc., Providence, RI, 2010. 317 pp. ISBN-10: 0-8218-4761-9. ISBN-13: 978-0-8218- 4761-9. Available from http://www.ams.org/bookstore-getitem/item=mbk-71.

[§] Selected Passages From Correspondence With Friends. ISSN 2054–7145. `http://www.borovik.net/selecta/`.