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THE ELEPHANT IN THE ROOM: THE ROLE OF INTUITION IN EVERYDAY MATHEMATICAL WORK

ALEXANDRE BOROVIK

In Memoriam
Cadbury the Netherland Dwarf,
my Easter Bunny[†]

Introduction

The aim of this report is to start a dialogue between research mathematicians and philosophers of mathematics on practical aspects of the use of intuition in the mathematical practice.

The present report describes my experiments with mathematical intuition and with the subconscious, which I did on three occasions. The first one was during lockdowns in July–December 2020 and was reported in my paper [4] and is briefly discussed in Section 1. The other two, documented in Sections 2 and 3, were associated with, and triggered by, the The Nancy-Liège Workshops on Mathematical Intuition: Nancy Edition, 30–31 May 2023.

The genre of the text is unusual: contemporaneous notes with some comments added later. In the English legal tradition, *contemporaneous notes* are notes made at the time or shortly after an event occurs and represent the best recollection of what you experienced or witnessed; in England, they are accepted in courts of law as documents. In this text, fragments taken from contemporaneous notes are clearly dated and set in a sans serif font.

I wish to emphasise that the present text is not yet a proper paper, but a backward reference for a shorter and more precisely formulated paper that I am currently preparing. Still I hope that it is sufficiently interesting on its own.

My narrative is relaxed and informal because it reflects an informal attitude to thinking common amongst mathematicians for whom mathematics is part of life, like breathing. Writing this text, I discovered that in my older papers I occasionally used the words *intuition* and *subconscious* almost interchangeably, which makes me slightly uncomfortable now. I leave refining their meaning to specialists. For me personally,

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[†] Cadbury unexpectedly died this year in the early hours of Good Friday.

- Doing mathematics involves a dialogue, two-way exchange of information with the subconscious.
- Intuition is the judgement arising from that dialogue.
- Or, maybe, intuition is the complex of mental skills needed for extracting meaning from the dialogue with the subconscious.
- But I am also prepared to accept that what I call intuition is completely different from what philosophers call intuition. But then what it is?

I am not an expert in philosophy or in cognitive science—I am satisfied with my labelling of what is happening in my head for as long as it helps me in my work in mathematics — and helps me to explain to my collaborators what I am talking about. I leave it to others to work on better formulations.

I have to emphasise that I do not consider myself being a kind of an exceptional case—there are, I am quite confident, tens of thousands of mathematicians like me around the world. I personally know a number of them.

1. *Lockdown, 2020–21*

Since my informal, but sufficiently detailed, report was published in [4], I give here only some quotes from it essential for understanding the present paper.

During the lockdown, as soon as teaching online, etc. ended in June 2020, I felt that I had suddenly become free and decided to start an experiment of sticking to a peculiar style of work and life.

My approach to work during the lockdown was perhaps bizarre, I quote from [4]:

> After a light breakfast and coffee, I start doing mathematics, that is, I sit at my desk and look out of the window. This is a hard job, and I soon become tired, move to a sofa and take a nap. On waking up, I am refreshed, and return to mathematics—and more often than not I have some new ideas for my work; they came to me during my sleep. This cycle is repeated.

The most important part of my work was done in the sleep—this is explained in [4] in some detail. Importantly, this work resulted in a decent paper [5]. My method: to nap several times a day and wait for ideas to emerge at awakening—was working.

I warned, however,

> ...any mathematics produced by a method so irrational as listening to the subconscious must be immediately, thoroughly, and rigorously checked. Trust your intuition – but remember that
>
> *verification is the highest form of trust.*

And I added

> ...to retain sanity and the necessary level of confidence in your work, help of colleagues could be useful and even necessary—someone else should take a close look at least at the key arguments in your paper.

I was lucky that even in lockdown I could rely on this help from my distant friends. This is why I started my paper [5] with an epigraph from John Donne (1623):

> *No man is an Iland, intire of itselfe; every man*
> *is a peece of the Continent, a part of the maine.*

My rabbit Cadbury played an important role in my life during the lockdowns. I quote [4]:

> [It is] one of the best kept secrets of mathematics: mathematics is done in the subconscious.
>
> A mathematician has to maintain good relations with his or her subconscious. The subconscious is not a properly domesticated beast, but it responds well to attention and kindness. It is like our rabbit, Cadbury the Netherland Dwarf (one of the wilder breeds of pet rabbits). When he is in good spirits, Cadbury grooms me, combing with his incisors the skin on my arm, apparently trying to relieve me of my (non-existent, I hope) fleas—this is a natural social behaviour of rabbits. While I doze, my subconscious combs the deepest recesses of my memory for morsels of mathematics which could be relevant to, or just somehow associated with the mathematics that I am trying to do in my conscious state. The subconscious is a wordless creature and brings its catch to the surface as a kind of uncertain, instantly disappearing visual image akin to a single frame inserted in a film reel (the "inverse vision" as described by William Thurston [13])[†]. Then another miracle happens: someone or something else in my mind looks at the catch and says: "well, this is. . . "—and gives the name, usually an already well-known term of mathematical language.

Returning to the adage 'verification is the highest form of trust', more can be said:

> Verification trains intuition, gives feedback to intuition, to the subconscious. If not properly supported by proofs and calculations, intuition wilts and dies. This simile in a more developed form can be found in my paper [3, Section 6.1], where the training of the subconscious is compared to a training of a dog.

2. *Five days, 29 May – 2 June 2023*

This section contains the contemporaneous notes, with some later comments, of my mathematical work during the five days, 29 May – 2 June 2023. I apologise that I avoid giving any details of the mathematics involved: the stuff is very technical.

Dateline

Day Zero: 21 May 2023 I received an email from a philosopher colleague, Brendan Larvor, with information about the Nancy Workshop on Mathematical Intuition 30–31 May. I think I registered as a participant and did not undertake anything else.

Day One: 29 May 2023

> I received a long, about 40 minutes, phone call from Adrien Deloro, a friend and co-author of mine. A few weeks before I made some comments on some research project suggesting that a certain shortcut could be used. Adrien explained me the reasons why the proposed shortcut would not work.[‡] Our conversation was a classical loose chat between two mathematicians, it contained no clear formulation of anything, but we understood

[†] This agrees with Amalric and Dehaene [1]:

> *By scanning professional mathematicians, we show that high-level mathematical reasoning rests on a set of brain areas that do not overlap with the classical left-hemisphere regions involved in language processing or verbal semantics. [. . .] Our results suggest that high-level mathematical thinking makes minimal use of language areas and instead recruits circuits initially involved in space and number.*

It is worth noting that Marie Amalric delivered a brilliant talk at the Nancy Workshop.

[‡] Our exchange of ideas was quite in the spirit of the ones described in Appendix C.

what we were talking about, and were directly appealing to our intuitions. Adrien is known for the exceptional clarity of his analysis of various situations in mathematics, and on this occasion this also was the case, even if our chat was absolutely informal.

I accepted Adrien's counterarguments, but said that I would continue to think—because I started to feel that my brain had gained momentum. As far as I remember, I did even stated, in the same loose terminology, what I would be thinking about —and Adrien understood what I meant. Let us call the object of my tentative contemplation the *Project*.

As soon as we finished our conversation, I found in my inbox an email from the organisers of the Nancy Workshop with the programme of the meetings next day, and abstracts of talks, and started to study them. Distracted by some other stuff as well, I did not do any mathematics that day – but I simply recalled, from time to time, that I have to do something with the Project.

Day Two: 30 May 2023

The first day of Workshop. I woke up in the morning with a clear understanding how certain obstacles pointed out by Adrien Deloro could be circumvented, and, moreover, that the desired result had to be formulated in simple and elegant terms—but I still had no appropriates words for this formulation. This demanded further thinking, but I had no time for that because I was watching, over Zoom, the (very interesting!) presentations on the Workshop.

I also emailed to my old friend and co-author Gregory Cherlin:

> *[T]his Workshop on Mathematical Intuition [. . .] is one of those cases when [I feel as if] I am a specimen under study. I routinely use intuition, every day, in my mathematical work. My co-authors and colleagues in my circle use intuition, we share our intuitions in conversations between us conducted in a remarkably loose and informal language which has nothing in common with the language of mathematics in print, we try to show the working of intuition to our PhD students and instruct them to develop their own intuition about particular mathematical concepts and entities.*

I illustrated my lament by inserting in the email the old *The New Yorker* cartoon by Leo Cullum, Fig. 1.[†]

For copyright reasons, the cartoon cannot be placed here. Instead, the reader is advised to make a search on Google Images for "I'm right there in the room, and no one even acknowledges me."

Figure 1. The New Yorker *cartoon by Leo Cullum (1942–2010)* I'm right there in the room, and no one even acknowledges me.

Gregory Cherlin's response came the same day:

> *I recall at the Newton Institute I got a key idea by looking at the solution to a chess problem I happened to see in a newspaper.*
> *I still recall seeing the bishop move (one I did not find myself, most likely) and thinking, "Of course, I can certainly do something like that." On some level this is nonsense but it is all that actually reached the conscious mind.*

[†]I discovered it in my old blog A Sentimental Journey a post of 21 May 2008, *Returning from a conference on philosophy of mathematics. . .* , where I have already expressed similar feelings:

> *. . . I am still full of this eerie feeling: a speaker after speaker was talking about mathematics and mathematicians, but, in their descriptions, I could recognise neither mathematics, nor myself. I was non-existent in the spooky world of philosophy.*

[...]

Possibly if [the subconscious] had been called the "superconscious" that would have helped [in a dialogue with philosophers]. But it is more of a distributed system—the mental network might be a better description.

I suppose at the level of terminology the point is that **the "conscious mind" is the narrative fragment—the journalist—not higher or lower than anything else, but always slightly behind the flow of events, and trying to keep up.** (Emphasis is mine—AB)

A later comment: I agree with Cherlin's last point. As I have already said in [4] (already under influence of his ideas—we had a number of conversations on that matter),

> What we perceive as the stream of our consciousness is memory, rationalization, and ex post facto justification, of decisions and actions already taken, and moreover, started by our subconscious. [...] We (in the sense of "conscious us") do not make decisions, we justify them retrospectively. Mathematicians, as it happens with people with a not very clear conscience, have a very sophisticated method of self-justification—construction of a proof. But even this is done in the same dishonest manner: when we exclaim something like: "So this should follow from the lemma about the three commutators!"—our subconscious has already known this for a long time and only sighs when hears our joyful cry.

However, now I find Cherlin's description of the stream of consciousness in mathematical thinking:

> *the "conscious mind" is the narrative fragment—the journalist—not higher or lower than anything else, but always slightly behind the flow of events, and trying to keep up*

more revealing and useful than my previously quoted rather emotive formulation [4].

But I return to my contemporaneous notes.

> When writing his email, Gregory knew that I knew what were the particular problem and the proof he was talking about. I remember a beautiful twist in the proof that he had suddenly produced.

I sent a similar email to Brendan Larvor, and recieved a response encouraging me to write down my experiences:

> *Convicts who have written about their experiences in prison and women who organise for better childbirth are welcome at conferences on criminology and obstetrics. Hearing the voice of the native has been an imperative for decades.*

In this report, I follow Brendan's advice; I am taking it very close to my heart because an additional (not mentioned in Section 1) aim of my lockdown experiment as described in [4] was checking whether mathematics can be done indeed in prison conditions, without any access to books, journals, websites, etc., on mathematics. I was guided by my own maxim that I formulated years ago, in my undergraduate years:

> mathematics can be done with a matchstick on a moldy wall of a prison cell.

—but please read [4] for details. Lockdown was indeed an appropriate proxy for incarceration albeit very comfortable.

Day Three: 31 May 2023

The second day of Workshop. I woke up in the morning with a clear understanding what the formulation should be — in a mathematical language different from the one Adrien and I were using before, and which also required a change in the point of view.[†] During coffee breaks in the Workshop, I wrote a page and half of a proof in LaTeX and sent a pdf file to Adrien, asking him whether it made any sense.

Day Four: 1 June 2023

Adrien responded to me, by text via WhatsApp,

"At first glance, cruelly make sense."

But I again woke up in the morning with new ideas, this time with understanding of how some general strategy could be combined with the new point of view, significantly advancing this specific direction of research.

Day Five: 2 June 2023

No new ideas — I simply had to check my proofs written the previous day and email to Deloro a four pages pdf file with these proofs and suggestions for further developments.

Adrien Deloro confirmed that all that made sense, and that these were serious results.

Discussion The following has been written before the end of June 2023 and is left as is:

Another story where I describe my games with the subconscious is [4].

I have already several times mentioned in print my simile of training the subconscious as training a dog — it could be found, for example, in [3, Sections 6.1–6.3].

I wish to reiterate that I am not something exceptional — there are, I think, tens of thousands of mathematicians like me around the world.

However not all mathematicians would share my views. I once ran a straw poll, with the question: "Do you think that mathematics is done in the subconscious?" among mathematicians who subscribed to a poetry mailing list, a kind of "A poem a day", which I administered during Covid lockdowns. The score was something like 55% *yes* to 45% *no*. This was not a scientific observation because the sample was likely to be skewed: these were mathematicians who loved poetry. I bet no sociologist or psychologist has ever studied this particular group of people: mathematicians who love poetry. Another warning signal: pure mathematicians were predominantly *yes* people, while applied mathematicians were mostly *no* people. Perhaps this is just one of many cultural differences between the two research communities.

28 June – 11 July, 2023: Abstract for the next Workshop. And this is an abstract for a talk *Intuition as seen by a working mathematician* which I submitted, online, on 28 June 2023, to *Liège Workshop on Mathematical Intuition*, 6–8 November, 2023—after a reminder of the approaching deadline of July 1 forwarded to me by Brendan Larvor on 21 June:

Intuition as seen by a working mathematician

As a working mathematician, I, my co-authors, and colleagues in my circle (those

[†] I used some elementary and well-known stuff from the Galois theory.

who I regularly talk with about my work) belong to a sizable community of mathematicians who

- Routinely rely on intuition in their research.
- Talk to each other in a loose informal language which has nothing in common with the language of mathematics in print.
- In these conversations, appeal to conversant's intuitions rather than to established formal facts.
- Convinced, from their experience, that mathematics is done in the subconscious.

I have already argued elsewhere that some aspects of mathematics could be seen as a language for communication with the subconscious, and that communication with the subconscious is the key aspect of mathematics.

Moreover, I described communication with the subconscious akin to training a dog and reading its signals. I would further argue that mathematician's intuition is the masters' model of expected behaviour of that dog and includes their shared emotions, while proofs have deep personal meaning: they are feedback to the subconscious to encourage, and reward it, for being correct in production of concepts, formulations of results, and directing proofs.

I will illustrate my theses by an example of my work over 5 days on and around the Nancy Workshop, I documented my work. I hope the story could be of interest to philosophers: it shows how intuition works in practice.

I hope my observation could be interesting to researches in philosophy of mathematics and cognitive mathematics. ■

On 11 July 2023 I received a very friendly rejection email from the organisers. I expected of course this turn of events because the page for abstract submission on the Workshop's website did not provide room for research mathematicians, Fig. 2.

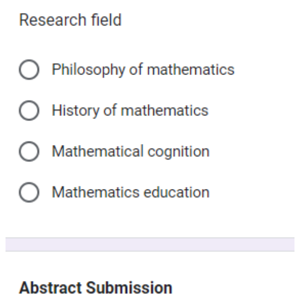

FIGURE 2. *Research fields: no room for mathematics as such.*

I do not remember which radio button I have clicked—technically speaking, I have some (minor) publications in all four fields.

3. *Procrastinational mathematics: Filling the spaces between the Fibonacci numbers 3, 4, 11, and 20 July 2023*

The only Ig Nobel prize for work of philosophical nature was awarded in 2011 (under the Literature category) to John Perry of Stanford University for his *Theory of Structured Procrastination*, which states:

> "To be a high achiever, always work on something important, using it as a way to avoid doing something that's even more important" [10, 11].

See also
Structured Procrastination, http://www.structuredprocrastination.com/.

In this section, I analyse an episode of my mathematical procrastination, that is, doing some silly elementary mathematics instead of what I really had to do.

Day Zero: 3 July 2023. A blunder

My friend and co-author Ayşe Berkman emailed to me:

> *We are both giving talks tomorrow at the Ukrainian Algebra Conference. Your title is Black Box Groups but your abstract is about Permutation Groups of FMR. If the latter is correct can you send your slides to me, so I do not have to repeat things.*

Yes, it was my blunder, and I could fix it only by giving the *Black Box Groups* talk[†]—and I had to give it next day. Fixing your own blunders in an unpleasant task; not surprisingly, I started to procrastinate—with a surprising result:

> **In my procrastinational doodling, I was apparently subconsciously directed to the same underlying structure as in my serious mathematical work described in Sections 1 and 2.**

I describe my venture into procrastinational mathematics with all mathematical details because the mathematics involved, if you do not dig too deep, is at the undergraduate level thus accessible to a wider audience, in contrast with the much more serious stuff mentioned in Sections 1 and 2.

Day One: 4 July 2023. Lazy doodling

> A colleague of mine wrote a book where he labeled chapters by Fibonacci numbers and then run into a not well-posed mathematical problem when he wished to insert another chapter into his book. I do not know what was his solution. But it so happened that this morning (4 July 2023), while writing a conference talk which I had to deliver two hours later[‡], I suddenly started to procrastinate by thinking about this problem.
>
> Being a mathematician, I started, of course, with an 'infinite' version of the problem.
>
> > Assume, for simplicity, that we have a book with infinitely many chapters labelled by the Fibonacci numbers F_n, $n = 0, 1, 2, \ldots$:
>
> $$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$
>
> The first two chapters get the same number 1. So what? The inimitable (and highly influential in the Western literature[§]) Laurence Stern's book *A Sentimental Journey through France and Italy* contains two consecutive chapters with the same title:
>
> THE REMISE[¶] DOOR.
> CALAIS.

[†] Based on a joint work with Şükrü Yalçınkaya, my friend and co-author of many years.

[‡] This was the talk at the Ukraine Algebraic Conference, nominally held at the city of Sumy in the north-east Ukraine, but conducted entirely on Zoom: Sumy was dangerously close to the front line, and under constant missile and shell fire.

[§] In Russia, Stern directly influenced Nikolay Karamzin, and Alexander Pushkin was a fan of Stern. Russian literature, as it is known in the West (and this includes Gogol, Dostoevsky, Turgenev, Tolstoy, Chechov), started with Karamzin and Pushkin.

[¶] The word *remise* is quite archaic; in my copy of *The Shorter Oxford English Dictionary*, published in 1955 (and containing 2515(!) pages) one of its meanings was *coach-house*, which exactly what it meant in Stern's narrative: for two chapters, the narrator of the story was chatting with a pretty lady in front of the couch house gate in Calais.

But still, it is perhaps advisable to avoid duplicating chapter numbers.

So, we have a question:

> What is the *natural* way to insert an integer number G_n strictly between F_n and F_{n+1}, that is, $F_n < G_n < F_{n+1}$?

It is convenient to include into the sequence the term $F_0 = 0$, then the Fibonacci sequence F is determined by the *initial values*

$$F_0 = 0, \qquad F_1 = 1$$

and the *recurrent relation*

$$F_{n-2} + F_{n-1} = F_n \quad n \geqslant 2.$$

It is well known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi,$$

the *Golden Ratio*.

The point F_{n+1} divides the segment $[F_n, F_{n+2}]$ in a similar fashion, and in proportion that also converges to ϕ:

$$\frac{F_{n+2} - F_{n+1}}{F_{n+1} - F_n} = \frac{F_n}{F_{n-1}} \to \phi \text{ as } n \to \infty.$$

Taking into account the sacral status of the Golden Ratio, it is worth trying to insert, between F_n and F_{n+1}, an integer G_n dividing the segment $[F_n, F_{n+1}]$ in an increasingly good approximation to the Golden Ratio:

$$\frac{F_{n+1} - G_n}{G_n - F_n} \to \phi \text{ as } n \to \infty.$$

The solution is obvious: take

$$G_n = F_n + F_{n-3},$$

then

$$F_{n+1} - G_n = (F_{n-1} + F_n) - (F_n + F_{n-3}) = F_{n-1} - F_{n-3} = F_{n-2},$$

and

$$G_n - F_n = F_{n-3},$$

so the segment $[F_n, F_{n+1}]$ is divided in the proportion

$$\frac{F_{n+1} - G_n}{G_n - F_n} = \frac{F_{n-2}}{F_{n-3}} \to \phi \text{ as } n \to \infty.$$

Moreover, it is easy to check that, for all $n \geqslant 4$,

$$G_n + G_{n+1} = G_{n+2},$$

so the sequence $\{G_n\}$ obeys the same law as the Fibonacci sequence $\{F_n\}$, and the two sequences are different because their initial values (at $n = 0, 1$) are different. □

A later comment: At that point I stopped further elaborations (and terminated my procrastination) even if the picture that I got was not yet to my satisfaction (and some aspects were even disturbing)—I had more important things to do. In the week of 3 July, I participated, and gave talks, in two mathematical conferences, and was busy. I almost forgot about my Fibonacci musings.

Day Two. 9 July 2023. Clarity

On a quiet Sunday morning of 9 July, I woke up with a complete explanation of my concerns, first of all, the irritating behavior of G_n for small non-negative values of n. Of course, it is difficult to insert an integer in strictly between 1 and 2, or 2 and 3. But for bigger n everything was so nice and smooth.

And the answer was: for the geometric sequence $\Phi = \{\phi^n\}_{n\in\mathbb{Z}}$, with indices spreading over all integers from $-\infty$ to ∞, we always have

$$\phi^{n-1} + \phi^n = \phi^{n+1},$$

and the points $\gamma_n = \phi^n + \phi^{n-3}$ accurately divide the segments $[\phi^n, \phi^{n+1}]$ in the ratio exactly ϕ.

I realised, that in my procrastinational doodling with the Fibonacci sequence, I somehow expected that G_n satisfied the Fibonacci Law, because I had, somewhere in the back of my mind, the understanding that I was working in the 2-dimensional real vector space $\mathcal{F}$ spanned by the sequences $\Phi = \{\phi^n\}_{n\in\mathbb{Z}}$ and $\Psi = \{\psi^n\}_{n\in\mathbb{Z}}$, where ϕ and ψ are the two roots of the equation $\lambda^2 = \lambda + 1$. It is easy to see directly from the Viéte Theorem (without using the quadratic formula) that one of the roots is positive (and we call it ϕ), and another is negative, and we call it ψ; $\phi > 1$ and $|\psi| < 1$.

So, $F = A\Phi + B\Psi$, and $F_n = A\phi^n + B\psi^n,\ \ n \in \mathbb{Z}$ for some real numbers A and B. The initial values $F_0 = 0$ and $F_1 = 1$ means that

$$A \cdot 1 + B \cdot 1 = 0$$
$$A \cdot \phi + B \cdot \psi = 1$$

which easily leads to

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}.$$

The vector space $\mathcal{F}$ could be equivalently defined as the space of all real valued sequences $H = \{H_n\}_{n\in\mathbb{Z}}$ satisfying, for all $n \in \mathbb{Z}$, the recursive identity

$$H_{n-1} + H_n = H_{n+1}.$$

It is easy to see that if two adjacent terms in H are integers, than all terms in H are integers. It is also obvious that the space $\mathcal{F}$ admits the shift operator $\boldsymbol{S}$ which moves terms in every sequence H:

$$\boldsymbol{S} : H \mapsto H'$$
$$H_{i+1} \mapsto H'_i$$

Obviously, $\boldsymbol{S}$ is a linear operator on $\mathcal{F}$ and the sequences $\{\phi^n\}_{n\in\mathbb{Z}}$ and $\{\psi^n\}_{n\in\mathbb{Z}}$ are its eigenvectors with eigenvalues, of course, ϕ and ψ.

But the Golden Ratio ϕ is irrational (this was known to Euclid, with a remarkable proof by infinite decent which uses nothing but the definition of the Golden Ratio). Since $\phi\psi = -1$ by Viéte, the root ψ is also irrational. Therefore the behaviour of integer sequences in $\mathcal{F}$ could be only some approximation of the behaviour of the '*eigensequences*' Φ and Ψ.

Day 3. 11 July 2023. Beauty and the Beast

I did not do any further work, and this morning simply add some comments.

First of all, it is very easy to expand the Fibonacci sequence F (or any sequence in $\mathcal{F}$) to the left, through negative indices:

$$\ldots\ 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The signs of the terms in F start alternating, and it is easy to see that

$$\lim_{n\to-\infty} \frac{F_{n+1}}{F_n} = -\phi^{-1}.$$

But

$$-\phi^{-1} = \psi,$$

the second eigenvalue of the shift operator $\boldsymbol{S}$. One may say that the two eigenvalues are twins, ϕ is good and ψ is evil, but I would compare them to the two faces of the Roman god Janus: $\phi > 1$ is positive and looks at the progress of F_n to the future $n \to \infty$, while $\psi < 0$ is negative and looks to the rather turbulent past.

I leave it as an exercise to the reader to check, that, for all $n \in \mathbb{Z}$, we have

$$\frac{G_n + G_{n-1}}{2} = F_n,$$

that is, F_n is exactly the midpoint of the segment $[G_{n-1}, G_n]$, so the two sequences F and G are very nicely placed with respect to each other.

My final comment: the answer to the doodle problem which started my story is

$$G = (\boldsymbol{S} - \boldsymbol{S}^{-3})F.$$

The shift operator $\boldsymbol{S}$ is of course invertible. This was why I instinctively started to think about sequences extending both ways, to the left and to the right.

Day Four. 19 July 2023. Further discussion

I somehow had all this understanding in my head, and my subconscious had access to it even during my sleep. However, I learnt that stuff, bit by bit, starting from my secondary school, and very rarely used it, and I did not expect how well connected these bits happen to be in my brain, in effect forming a working mechanism—just push the button.

Is that what philosophers call intuition?

Keith Devlin [8] once coined the expression[†]

Mathematics is the science of patterns.

It became quite popular among school teachers of mathematics and popularises of mathematics and our colleagues writing about mathematics [9]. The Fibonacci sequence and the Golden Ratio were of course used as an example of beautiful and important patterns. I hope that my notes explain that actually

Mathematics is the science of structures behind patterns.

After writing that I realised that I did not know how to describe to lay people (that is, those who did not have a proper mathematics education) what I saw behind the Fibonacci sequence. I was busy with something else and decided to postpone this task—thus acting in the best British tradition immortalised in the old comedy *Carry On . . . Up the Khyber* (1968)

Captain Keene : [news of the native revolt arrives] What do you intend to do, sir?
Sir Sidney Ruff-Diamond : Do? Do? We're British. We won't do anything. . .
Major Shorthouse : . . . until it's too late.
Sir Sidney Ruff-Diamond : Exactly. That's the first sensible thing you've said all day.

Day Five. 20 July 2023

As soon as I started to think about the structure behind the Fibonacci and other linear recursive sequences I was struck by a kind of mental lightning, an instant revelation, a crystallisation of a *Gestalt*, a jolt through my brain.[‡] A very interesting emotion, I experienced it many times before, first time perhaps at the age of 6, when I suddenly realised that I could read—when in a split second combinations of letters on paper became alive, got meaning, were expressing emotions. [See Appendix A.]

The following paragraph of my contemporaneous note of 20 July 2023 was slightly

[†] As Brendan Larvor explained to me, it seems to originate with Michael Resnik [12], but Devlin really popularised it.

[‡] It is like being lost in a forest or an unknown city and after some wandering around you suddenly realise that you have reached an already well know to you place, but approached it from an unusual direction—and you instantly see, in your inner vision, a mental map of locations already known to you.

edited on 27 July 2023, mostly for aesthetic reasons. Changes are shown by using a serif font, the original version is in Appendix B.

> This structure consisted of
> - A vector space V over some field K, perhaps infinite dimensional, while K could be finite.
> - A group Σ of invertible endomorphisms $\sigma \in \operatorname{End} V$ on V.
> - Let us call a Σ-invariant subspace $U < V$ *bounded*, if for every $\sigma \in \Sigma$ there is a non-zero polynomial $p \in K[x]$ such that $p(\sigma) = 0$ on U.
> - The union $V^\flat$ of all Σ-invariant bounded subspaces $U < V$.

In particular, if Σ is periodic, then every $K\Sigma$-module is obviously bounded: for $\sigma \in \Sigma$ of order n, the polynomial $\sigma^n - 1 = 0$ on V. Also, by the Cayley-Hamilton Theorem, every finite dimensional K-vector space is bounded with respect to the action of any group Σ on it.

For $\sigma \in \Sigma$ and U is bounded with respect to Σ, we denote by $\mu_U(\sigma)$ the monic polynomial $q(x) \in K[x]$ of minimal degree such that $q(\sigma) = 0$ on U; it is unique and in the basic linear algebra known as the *minimal polynomial* of σ—but the vector space U is not necessary finite dimensional over K.

And now I return to my contemporaneous note of 20 July 2023.

> In the case of the Fibonacci sequence, all that is an elementary stuff, and, I gather, can be found in any book on linear recursive sequences (but I did not look in books on linear recursive sequences for at least 40 years). Here, $V = \mathbb{R}^{\mathbb{Z}}$, and the endomorphism σ is the shift operator $\boldsymbol{S}$ which generates the cyclic group $\Sigma \simeq \mathbb{Z}$, but we focus on σ itself. If
>
> $$\tau = \sigma^m + a_{m-1}\sigma^{m-1} + \cdots + a_1\sigma + a_0, \quad a_i \in \mathbb{R},$$
>
> then $\mathcal{U} = \operatorname{Ann}_V(\tau)$ is the (finite dimensional!) vector space of sequences $U = \{U_n\}$ defined by the linear recursive relation
>
> $$U_{n+m} = -a_{m-1}U_{n+m-1} - \cdots - a_1U_{n+1} - a_0U_n$$
>
> and
>
> $$\mu_{\mathcal{U}}(\sigma) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$$
>
> is the minimal polynomial of σ (or, what is the same, of $\boldsymbol{S}$) on $\mathcal{U}$.
>
> I apologise, all these explanations are perhaps not for lay people. Still, this was the situation in which I found myself.
>
> What really astonished me is that my sudden observation was [was that the structure created in (or maybe directing?) my procrastination doodling was]
> - the same structure that I reified in my work during lockdown described in [4] and which resulted in my paper [5]; here, the field K was finite and the vector space V infinite, and the finite group of endomorphisms Σ had nothing in common with the shift operator $\boldsymbol{S}$ on $\mathbb{R}^{\mathbb{Z}}$. [Added on 1 August 2023: In the proof of [5, Theorem 4] Σ was an infinite periodic group.]
> - It was quietly present in my conversation with Adrien Deloro, and some its specification was critical in my observations resulting from this conversation; here, again, the endomorphism σ had nothing in common with the shift operator in the theory of linear recursive sequences.
> - [Added on 27 June 2023: Earlier, Adrien and I came to conclusion, that, in group actions on vector spaces as they arise in model-theoretic algebra, the concept of the character value of a group element makes no sense, but the minimal polynomial makes sense for *bounded* representations. In the case of linear recursive sequences, the minimal polynomials of the shift operator become the central concept of the theory.]

- And I would mention it or at least quote some results from [5] in my originally planned talk at the Sumy Algebra Conference.

So when I had to write slides for a talk which was different from the originally planned, and suddenly started to procrastinate over a silly elementary problem about Fibonacci numbers it was, it seems, my mind's resistance—it wished to stay on a familiar territory.

It seems that the subconscious easily handles high level concepts and structures and is happy to work at the highest available level of abstraction—*the higher the happier*.

And finally, I have to ask Adrien Deloro whether the answer to the following question:

> Describe infinite groups which have faithful irreducible bounded representations by endomorphisms of infinite dimensional vector spaces over fields from a particular class (say, finite, pseudofinite, algebraically closed)?

could be useful in the model-theoretic algebra?

Actually, quite a number of questions could be asked about bounded representations. Perhaps, they have been already studied, under different name.†

I would not be surprised, if the answer to this question:

> Is it true that if a compact Lie groups has a faithful irreducible unitary representation which is bounded than it is finite dimensional?

was already known, in a different terminology, for about a century.

Given an action of a group Σ on a K-vector space V, we can call an element $\sigma \in \Sigma$ *bounded* in this representation, if $p(\sigma) = 0$ on V for some polynomial $p \in K[x]$. Elements of finite order are bounded; also, by definition, elements which are "unipotent" in this representation. And what else?

Belonging to the part of mathematics feeding on the interplay of finite and infinite, boundness appears to be a pretty strong finiteness condition. I can easily continue the list of natural questions.

You would perhaps agree that mathematical procrastination could be productive and lead to some deep mathematics. This is an aspect of mathematical practice which perhaps has never been studied. It is my guess that a lot of mathematical folklore, say many mathematical games and puzzles were born of procrastination. The role of intuition and the subconscious in this peculiar psychological phenomenon is, in my opinion, quite fascinating.

02 August 2023. The end of the story about procrastination I finally decided to look for some references for my paper, and, in the search in the WIKIPEDIA discovered that the vectors space $\mathcal{F}$ of Fibonacci-type sequences contains another relatively famous sequence: *Lucas numbers* $L = \{L_n\}_{n\in\mathbb{Z}}$, defined by the initial values $L_0 = 2$ and $L_1 = 1$, sequence A000032 in The On-Line Encyclopedia of Integer Sequences (OEIS®). I have not heard about it before. By observation,

$$G = \boldsymbol{S}L \text{ or, if you prefer, } L = \boldsymbol{S}^{-1}G.$$

The sequence G is not contained in OEIS®.

---

†Boundness could be also called the *Cayley-Hamilton property*, for example.

I feel lucky that, in my procrastenation, I decided (or simply was lazy enough) not to look for any information in the existing literature—for otherwise I would not get some, I hope, useful insights for my mainstream projects.

4. *The role of the subconscious in error detection*

Finally, a short observations which I had already made in my short paper (actually, a letter to the editor) which have just appeared in *The Mathematical Intelligencer* [6]. It touches on the role if the subconscious in error detection.

Indeed, error detection has an interesting neurophysiological aspect. In words of Stanislas Dehaene, the world leading researcher in mathematical brain,

> '*the brain reacts to errors even if they are not registered by the conscious part of the mind*'. [7]

He explains further that the electromagnetic impulse signalling Error! originates in the *anterior cingulate cortex*, a region of prefrontal cortex located on the midline of both hemispheres. It is so powerful that could be easily detected by electrodes glued to the skin of the head —no invasive methods or tomography are needed. Still, in many cases the conscious does not register it.

This is one of the most powerful signals sent back by the subconscious to the conscious. In teaching mathematics, we have to help our students to learn to register and understand at least this signal—this is critically important for their further progress in mathematics.

In supervision of PhD students, it is crucially important to give *quick* feedback on their work. I frequently did it by scanning their texts diagonally; it was not 'speed reading', I did not try to read—my aim was to catch a sufficient number of most obvious inconsistencies, gaps in arguments, typos, all kinds of mistakes, so that their correction would make my student busy for a day or two. Some lines in the text simply jumped at me—I had to stop and think for a few seconds to start seing the error. My subconscious detected a possible error before I even was able to understand what the suspect line was about. Of course, it does not mean that this approach allows to detect *all* errors—of course not! This is why this routine had to be repeated several times before the manuscript reached a readable form for a serious in-depth reading.

And have a look at Appendix C—it contains discussion of situations when mathematicians almost instantly identified that a result communicated by a colleague in a conversation was likely to be wrong.

5. *Conclusion*

I hope philosophers of mathematics accept that professional research mathematicians also have some relation to mathematics, and their views on mathematics perhaps deserve a kind of registration or maybe even acknowledgement. I plan to write a shorter text, with less mathematical detail, and no formulae, with a more compact formulation of my position.

Appendix A. Learning to read

This is a fragment from my paper [2].

Figure 3 shows the cover of the first book that I read in my life. Moreover, it was

FIGURE 3. *V. Suteev,* Under the Mushroom.

the book that had taught me to read, and it remains the most powerful intellectual experience of my life.

By the age of 5 or 6 (school in Russia started from 7), I somehow learned the alphabet—mostly by asking my parents and my brothers (who were at the time about 14–15 and 10–11 years old). Crucially, my mother was an elementary school teacher and for that reason, I believe, letters were explained to me by the corresponding sounds, not by their names. No-one, however, taught me to read in any systematic way. For some time the relation between letters on paper and words remained a complete mystery to me. When my brother read me a book, I was very suspicious and occasionally complained to my parents that he was inventing the words.

Then one day I was sitting in a quiet corner with Vladimir Suteev's book Под Грибом. Looking at the picture of a mushroom ("гриб" in Russian) I suddenly realized that the letters

П, О, Д, Г, Р, И, Б, О, М

on the cover could mean only "Pod Gribom", "Under the mushroom", and nothing else. I remember I was surprised by my discovery and decided to check. Under closer examination, it was still

П-О-Д Г-Р-И-Б-О-М.

"So this is how they are doing that"—I thought to myself, then opened the little book and read it in one go, from the beginning to the end. Then I went to the kitchen to report my progress to my mother. "Mom"—said I,—"I read a book". "Really?"—said my Mom without any sign of surprise. To prove my point, I read the book to her aloud.

> Next day, I remember, I went to the village library and enrolled as a borrower.
>
> What I experienced was a classical "AHA" moment, with the same emotional charge as in mathematical discovery. I suddenly discovered that a parser for processing typed text got assembled in my brain and was ready for use. It was exactly the same feeling as in doing mathematics, many years later. ■

If you think that this story is an exaggeration, read [2] – it explains the advantages of synthetic phonics in learning to read. Of course, the Russian orthography is more phonetic than English, which makes synthetic phonics much easier to use.

Appendix B. Contemporaneous note of 20 July 2023

This is a fragment from the contemporaneous note of 20 July 2023 before it was edited.

> This structure consisted of
>
> - A vector space V over some field K;
> - an endomorphism $\sigma \in \operatorname{End} V$ on V;
> - the enveloping algebra E of σ, that is, subalgebra generated by σ in $\operatorname{End} V$,
> - For every E-invariant finite dimensional subspace $U < V$, a map
>
> $$\begin{aligned} \mu_U : E &\longrightarrow K[x] \\ \tau &\mapsto \text{ the minimal polynomial of } \tau \text{ on } U \end{aligned}$$
>
> - The union W of all E-invariant finite dimensional subspaces $U < V$.

It is replaced in the main text by a simpler and a bit more general formulation.

Appendix C. Correspondence between two mathematicians

This is a recent exchange of emails, in September 2023, between two very experienced mathematicians (Gadde Swarup and Gregory Cherlin), on the fascinating topic which has emerged several times in the present text: evolution of ideas in conversations between mathematicians. It was forwarded to me by Gregory Cherlin, who by that time had already seen an earlier version of the present text.

The texts of emails in the exchange are slightly redacted.

Some of the material is also going to appear in a book in Peter Scott's honor, but is in the public domain, having appeared in a post Reminiscences of Peter Scott by Gadde Swarup in his blog Gaddeswarup's blog, 14 September 2023.

1. An email from $\mathcal{GS}$ to $\mathcal{GC}$, 21 Sep 2023, 12:24 AM:

> Peter suggested the problem in Norman, Oklahoma in 1986 and had a solution ready. Right from the beginning I felt that he was wrong. He went off to Liverpool and I wrote to him about the mistake. But the problem started bugging me I thought that a paper by Cannon and Thurston may help. It gives a map from a limit set to the circle which I thought was finite to one and cannot be onto. I went to a conference and met Cannon and asked him and he said yes. It was still a silly idea because it was all set theoretic sort of idea. I worked hard on it for a year and finally made it

work. I think Peter moved to Michigan by that time and I went to discuss with him. He asked

‘Do you really want to make this a joint paper?’

I said yes. Another paper we discussed then, he did it completely. Then he too struggled through Cannon-Thurston and wrote it up. That was the first time I came to grips with some of Thurston’s work.

Two interesting occurrences that happened later. I told Bestvina proudly about it. He immediately said

‘Why don’t you do this?’

which was exactly what I did. When I told Thurston about it, Thurston seemed surprised. Then he looked up to the sky for ten seconds and said

‘Of course’.

It turned out to be a useful result.

2. From $\mathcal{GC}$ to $\mathcal{GS}$, 21 Sep 2023, 3:10 PM:

Nice story, mind if I send it to Borovik? He thinks a lot about this sort of thing (kind of mathematical epistemology).

By the way, $\mathcal{X}$ told a somewhat different story about $\mathcal{Y}$. $\mathcal{X}$ was telling $\mathcal{Y}$ his conjecture about [redacted] and $\mathcal{Y}$ just said

‘No, no, it’s not like that, not like that at all.’

Which is pretty intimidating. So after a pause $\mathcal{X}$ said,

‘Well, why not?’

So $\mathcal{Y}$ starts to think, lifts up a hand involuntarily behind his head, looks up toward the ceiling, and says,

“Can it beeeeeeeee?””.

I had a similar experience once (I mean, similar to $\mathcal{X}$) with another mathematician. Ending a bit less vigorously with

‘Well, maybe so.’

For some reason, maybe because of your last, or because I’ve been thinking about De Branges, I found myself thinking of how trivial the actual solution to a problem can be when one has figured out what the technical point is, something I’ve also experienced ”recently” (which nowadays I interpret as meaning a few years ago). And there is the step that took Wiles a year in fixing his proof, namely treating primes congruent to 3 before primes congruent to $1 \mod 4$ (or maybe the other way around, I don’t actually recall which way worked and which way didn’t).

Tracing the evolution of mathematical ideas is really hard. After all, everything is more or less a tautology when finished. So it’s very sensitive to the language used and the different intuitions that are coming into play around the same set of ideas. The historians like to argue about whether an idea is ”present” or not in a text. Sometimes at least we experience the passage from not understanding to understanding, on an individual level—whatever that means. It’s not so clear, especially if the point is just to repeat something you already know, but in the right context.

Mostly enjoyable, in any case. And more enjoyable when one has others to talk to. Though sometimes one can dig things out of papers (which is maybe easier now). Generally takes a long time preparing the ground, or planting the seeds, or just sitting around waiting for something to turn up.

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Gadde Swarup and Gregory Chelin kindly allowed me to use their exchange of emails (Appendix C).

Disclaimer

The author writes in his personal capacity and the views expressed do not necessarily represent position of his (former) employer or any other person, corporation, organisation or institution.

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About me

I am a research mathematician but I have 40+ years of teaching experience at secondary school and university level in four different countries with four different education systems; since 1998 I was a Professor of Pure Mathematics at the University of Manchester, currently I am a retired Professor Emeritus. I published 80+ papers and co-authored three monographs in hardcore mathematics.

I also have an interest in cognitive aspects of mathematical practice; see my book *Mathematics under the Microscope* ((1) in the list below), which explains a mathematician's outlook at psycho-physiological and cognitive issues in mathematics and mathematics education, and touches on some issues raised in this paper. Some of my papers on mathematics education, etc. can be found in my personal online journal/blog *Selected Passages From Correspondence With Friends*.

A few published papers and a book on issues around mathematics:

(1) Response to "'Decolonization of the Curricula: Beyond Historical Enrichment". *The Mathematical Intelligencer* 2023, https://doi.org/10.1007/s00283-023-10295-1.
(2) 'Decolonisation' of the curricula. *The Mathematical Intelligencer* 45 no. 2 (2023), 144–149. DOI `https://doi.org/10.1007/s00283-023-10269-3` Earlier version: arXiv:2212.13167v3 [math.HO].
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EMAIL `alexandre ≫at≪ borovik.net`
WEB www.borovik.net