

## 101 COINS

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### 1. *The Problem*

This paper discusses the following elementary mathematics problem:

101 COINS. You are given 101 coins such that after removal of any one of them the remaining coins can be redistributed in two groups of 50 coins in such a way that the sum of values of coins in each group is the same. Prove that all coins have equal values.

It has an elementary solution, but also happens to be surprisingly deep.

### 2. *A solution without a single formula*

What follows is a solution which could be explained to junior school pupils. Well, I have to insert a caveat: to junior school pupils who have a sufficiently long attention span (developed, for example, by building reasonably sophisticated models from Lego, or by coding in SCRATCH, or by learning some mathematics beyond school curriculum. ).

There is a good reason for this requirement. In the original formulation with coins, the problem is next to impossible for a schoolchild – it needs to be simplified first by *generalisation* and made more manageable by *nomination*, then approaches to a potential solution need to be found by observing *invariance* under certain *transformations* of the data, and, on the top of that all, we need to use *abstraction*. All these tools are routinely used in solving so-called mathematical olympiad problems – and in research level mathematics.

### 2.1. *Generalisation*

It comes first, and it is easier for children. First of all, 101 appears to be just an arbitrary odd integer. Why odd? Because if the number of coins is even, then after removal of one coin it becomes odd, and remaining coins cannot be divided in two groups with equal numbers of coins.

So we have infinitely many problems, one for each odd number. Even if we continue to use 101 in our solution, we have to take care of ensuring that 101 could be replaced by any odd natural number.

### 2.2. *Nomination*

Then comes *nomination*: we need a name for collections of coins appearing in these problems.

So let us call a collection of coins *peculiar* if it contains an odd number of coins and, no matter which coin is removed, the remaining coins can be divided in two groups which contain equal number of coins and also have equal sums of values of their coins.

Having a name for the property allows us to ask a question: what kind of manipulations with coins preserves this property: *peculiarity*?

### 2.3. *Invariance and abstraction*

After some guesses we can notice that if we decrease or increase the value of each coin by the same amount, for example, subtract 1 from the value of each coin, a peculiar collection of coins remains peculiar – an adult mathematician would say that the peculiarity is *invariant* under this transformation.

But this is a rather unnatural operation on real coins, it is known as *debasement of the coinage* and, I am afraid, is likely to be a criminal offence in this country<sup>†</sup>. So it is convenient to forget about coins and work with natural numbers. This act of *abstraction* suddenly gives us freedom of action.

So it is useful to remember this guiding principle:

ABSTRACTION IS FREEDOM.

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<sup>†</sup>The crime was popular in old times when coins were made of gold or silver, which led weaker souls to temptation to chip a little bit of precious metal off the coin.

After applying this tiny bit of abstraction our problem looks that way:

We are given a set of 101 natural numbers, and after removal of any one of them from the list the remaining numbers can be redistributed in two groups of 50 numbers in such a way that the sum of numbers in each group is the same. Prove that all numbers are equal.

Of course, it is natural to apply the term *peculiar* not only to sets of coins but also to sets of natural numbers and to sets of integers (and, in later discussion – to sets of real numbers).

#### 2.4. *First breakthrough*

And now we can make a critical observation:

after removal of any one number from a peculiar set of natural numbers the sum of the remaining numbers is even.

Would you agree?

Hence

in a peculiar set of integers, all numbers have the same *parity*: either

- (a) they are all even, or
- (b) they are all odd.

It is a significant progress: we know something new about peculiar sets. On a way to a solution, we started to develop a small theory, a toy model of mathematics.

#### 2.5. *One more transformation preserving peculiarity*

In case (a) – when all numbers in the set are even – we have one more transformation which preserves the peculiarity of the set:

*Division by 2*: if all numbers are even, divide them all by 2  
– the resulting set of numbers is still peculiar.

It makes sense to apply this operation every time we can: we get a new peculiar set with smaller numbers, hence a simpler problem.

In case (b) we can apply the already familiar

*Subtraction of 1*: if all numbers in the set are odd, subtract 1 from all numbers. Again, we get a new peculiar set, but

with smaller numbers – and all numbers in this new set are even.

## 2.6. *Impossibility of infinite descent*

So, given a peculiar set, we can make from it a new peculiar set with smaller numbers by applying either subtraction of 1, or division by 2.

If in the new set all numbers are equal, then in the original set all numbers were equal, too. So we found a way to *reduce* our problem to a problem with strictly *smaller* numbers. In mathematics, there is name for this reduction: *descent*.

And we can continue this process for ... Not forever; sooner or later one of the numbers becomes 0.<sup>†</sup> If numbers are all zeroes, we are done: this means that in the original set all numbers were equal.

So we need to exclude the possibility for a peculiar set to contain at least one number 0 and at least one non-zero number. Since 0 is an even number, all numbers in the set are even. Now continue with divisions by 2 (which do not change our number 0), until some number in the set becomes odd. But this is impossible, since 0 and this number have different parities.

This completes the proof. □

## 3. *Discussion*

### 3.1. *Compression of thought*

The proof can be compressed into one paragraph, and this is what was my original solution.

After removal of any one number the sum of the remaining numbers is even. Hence all numbers have the same parity: either they are all even, or they are all odd. In the former case, divide all numbers by 2. In the latter, subtract 1 from all numbers. In both cases we get the same problem, but with smaller numbers, and this continues until one of the numbers becomes 0. If now numbers are all zeroes, we are

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<sup>†</sup>This observation has a name in mathematics: the *Principle of Infinite Descent* (a better name would perhaps be *Impossibility of Infinite Descent*); it was formulated in modern terms by Pierre Fermat.

done. Assume that there is a non-zero number left. Now divide all numbers by 2 and continue that until some other number becomes odd. This is a contradiction: numbers now have different parities.

Unfortunately, in this compressed solution we lose most of the colours and shades of the delicate transformation of the problem.

When adult mathematicians solve relatively elementary problems like this one, their thinking is usually compressed; the words like “infinite descent” just flash in their brains for a split second, barely registered by the conscious part of mind. However, children have to solve their first problems step-by-step – but given support and freedom, they can quickly develop ability to compress their thinking. Actually, a tendency to rapid abbreviation, *compression of reasoning* in problem solving is a characteristic trait of the so-called “mathematically able” children<sup>†</sup>.

Discussion of the 101 COINS problem was started, in a circle of mathematician friends, by my good colleague Hovhannes Khudaverdyan. He formulated it in a more general form, when instead of values of coins (which are positive integers), we are given their weights (which can be arbitrary positive real numbers).

101 COINS, BY WEIGHT RATHER THAN BY VALUE. We are given 101 coins such that after removal of any one of them the remaining coins can be redistributed in two groups of 50 coins in such a way that the sum of weights of coins in each group is the same. Prove that all coins have equal weights.

Khudaverdyan found a solution based on manipulation with determinants which was simplified by James Montaldi by switching, at some crucial point, to calculation over  $\mathbb{Z}/2\mathbb{Z}$  – that is, with symbols 0 and 1 which represent “even” and “odd”; more on that later. To say the truth, I did not follow the discussion until Khudaverdyan emailed to us a solution proposed by Alexander Karabegov. My eye caught the phrase “*To avoid dealing with 2-adic numbers...*”. After that I wrote, on the spot, the one paragraph solution and emailed it to friends. For a mathematician, the word “2-adic” suffices. Here, I prefer not to go into 2-adic numbers into any detail, but still wish give a hint at what it means – see the next two Sections.

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<sup>†</sup>See A. V. Borovik and A. D. Gardiner, *Mathematical abilities and mathematical skills*, The De Morgan Journal 2 no. 2 (2012) 75-86. [bit.ly/2jTYy4r](http://bit.ly/2jTYy4r).

### 3.2. *Further compression of solution: base 2 arithmetic*

The 101 COINS problem (for natural numbers) is one of the cases when base 2 arithmetic is useful.

Indeed, solution of the 101 COINS problem is a hidden calculation in base 2 arithmetic: dividing by 2 is crossing out 0 in the rightmost position, subtraction of 1 is replacement of 1 in the rightmost position by 0: number  $7 = 111_{\text{base } 2}$  becomes  $110_{\text{base } 2} = 6$ , and  $6 = 110_{\text{base } 2}$  becomes  $11_{\text{base } 2} = 3$ .

As you can see, the solution becomes really short.

### 3.3. *2-adic integers*

And now a few words about *2-adic integers*: they are numbers written in base 2 which are allowed to go to infinity to the left, something like

$\cdots 1010101_{\text{base } 2}$  or  $\cdots 111111_{\text{base } 2}$  (in the latter, all symbols are 1),

with usual operations of (long) addition and (long) multiplication, but with unusual results: notice that

$$\cdots 111111_{\text{base } 2} + \cdots 0000001_{\text{base } 2} = \cdots 0000000_{\text{base } 2} = 0,$$

hence

$$\cdots 111111_{\text{base } 2} \text{ (all 1's)} = -1.$$

Every natural number is of course a 2-adic integer; for example,

$$7 = \cdots 00111_{\text{base } 2} \text{ (infinitely many 0's)}.$$

## 4. *The problem in the context of linear algebra*

And now we switch the point of view and look at the problem in the context of linear algebra.

### 4.1. *Generalisation to real numbers*

But what if numbers involved are not natural, but arbitrary real numbers?

First of all, if we have a peculiar set of rational numbers, we can multiply them all by their common denominator and turn them all into integers. Then we can add to them all a sufficiently large positive integer and make them all positive integers, that is, natural numbers.

This new set of numbers is still peculiar, hence all numbers in it are equal, hence all original numbers are equal.

The case of an arbitrary peculiar set of real numbers can be reduced to the rational case, the argument is very simple and short, but uses some abstract linear algebra which lies beyond skills of most graduates of mathematics departments of British universities. The following argument was produced by Alexander Karabegov.

Let  $n$  be an odd natural number and  $X = \{x_1, x_2, \dots, x_n\}$  be a peculiar set of real numbers. Consider the  $\mathbb{Q}$ -vector space spanned by  $X$  in  $\mathbb{R}$  and let  $B = \{b_1, \dots, b_k\}$  be its basis. Then each  $x_i$  can be written as a linear combination of  $b_j$ 's with rational coefficients,

$$x_i = \sum a_{ij} b_j.$$

Observe that, for each  $j$ , the rational numbers

$$a_{1j}, a_{2j}, \dots, a_{nj}$$

form a peculiar set, hence are equal. But this means that all  $x_i$  are equal.  $\square$

At a philosophical level, we deal with two completions of rational numbers: non-Archimedean (2-adic numbers) and Archimedean: real numbers. The link between the two via a basis in a vector subspace over  $\mathbb{Q}$  in  $\mathbb{R}$  strikes by its demonstrative, in-your-face, discontinuity: numbers in a peculiar systems of natural numbers obtained from

$$\{x_1, x_2, \dots, x_n\}$$

cannot be represented by continuous functions of  $x_1, x_2, \dots, x_n$ !<sup>†</sup>

#### 4.2. A matrix algebra solution: back to integers

Finally, without reduction to rational numbers and to integers, the proof can be done by matrix theory, by analysing eigenvectors and eigenvalues of the matrix of the system of simultaneous linear equations in unknowns  $x_1, x_2, \dots, x_{100}, \dots, x_{101}$  which stand for numbers forming a peculiar set. What follows is a proof developed by James Montaldi from the original proof by Hovhannes Khudaverdyan.

Denote the numbers in a peculiar set by  $\{x_1, x_2, \dots, x_{101}\}$ . The problem evidently is reduced to the following matrix problem.

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<sup>†</sup>I leave proof of this fact as an exercise to the reader.

First of all, write our peculiar set as a vector,

$$\mathbf{x} = (x_1, \dots, x_{101}).$$

Consider a matrix of the size  $101 \times 101$ ,  $M = (m_{ik})$ ,  $i, k = 1, \dots, 101$ , such that all diagonal elements of this matrix are equal to zero and in every row some 50 non-diagonal entries are equal to  $+1$  and another fifty non-diagonal entries are equal to  $-1$ . The peculiarity of the set means that signs  $+$  and  $-$  can be chosen in such a way that

$$\mathbf{x}M = \mathbf{0},$$

that is,  $\mathbf{x}$  becomes an eigenvector for  $M$  for eigenvalue 0.

Please observe that we returned back into the domain of integers: the weights  $x_i$  are real, but the coefficients of the matrix  $M$  are *integers*!

It is time to recall that it is evident that the vector

$$\mathbf{e} = (1, 1, \dots, 1)$$

is an eigenvector of the matrix  $M$  with eigenvalue 0. To solve the problem, we need to prove that the space of zero eigenvectors is 1-dimensional, hence  $\mathbf{x}$  and  $\mathbf{e}$  are collinear and therefore

$$x_1 = x_2 = \dots = x_n.$$

In other words, we need to prove that the rank of the matrix  $M$  equals to 100.

At this point the solution bifurcates into two different versions: calculation of the rank of  $M$  by analysis of its characteristic polynomial (original solution due to Hovhannes Khudaverdyan) and vector algebra over  $\mathbb{Z}/s\mathbb{Z}$  (improvement suggested by James Montaldi).

#### 4.3. *Back to arithmetic modulo 2*

Indeed, arithmetic modulo 2 comes into play in the following elegant argument found by James Montaldi.

Consider the matrix  $A = M + I$ ; all its entries are  $\pm 1$ . Now consider it modulo 2, treating entries as elements of the field of residues  $\mathbb{Z}/2\mathbb{Z}$ . The vector  $\mathbf{e} = (1, 1, \dots, 1)$  is an eigenvector of  $A$  for eigenvalue 1 over rational numbers, that is, eigenvalue 1 over  $\mathbb{Z}/2\mathbb{Z}$ . But any vector over  $\mathbb{Z}/2\mathbb{Z}$  orthogonal to  $\mathbf{e}$  is an eigenvector of  $A$  for eigenvalue 0 over  $\mathbb{Z}/2\mathbb{Z}$ ; furthermore this vector is an eigenvector of  $M = A - I$  with eigenvalue 1. Hence  $M$  has rank 100 over  $\mathbb{Z}/2\mathbb{Z}$  and therefore rank 100 over  $\mathbb{Z}$ .



4.4. *The characteristic polynomial of  $M$ : do we really stay in characteristic 0?*

Hovhannes Khudaverdyan's solution is based on the observation that a  $101 \times 101$  matrix  $M$  has rank 100 if and only if the multiplicity of its eigenvalue  $\lambda = 0$  equals 1, which, in its turn, is equivalent to the coefficient  $a_1$  for  $\lambda$  in the characteristic polynomial

$$\det(M - \lambda I) = a_0 + a_1\lambda + \cdots + a_{100}\lambda^{100} - \lambda^{101}$$

being not equal to zero. And this will be proven by showing, surprise-surprise, that this coefficient is odd.

To do that, we need a bit of combinatorics. A *derangement* of a set  $X$  is a permutation  $X \rightarrow X$  without fixed points. Let us denote by  $\mathcal{D}(X)$  the set of all derangements of  $X$ . We will be working with 101 sets  $X_i = \{1, 2, 3, \dots, 100, 101\} \setminus \{i\}$ , where  $i = 1, 2, \dots, 100, 101$ .

The definition of determinant gives us

$$a_1 = \sum_{i=1}^{101} \sum_{\sigma \in \mathcal{D}(X_i)} \text{sign}(\sigma) \prod_{j \in X_i} m_{j, \sigma(j)}.$$

Observe that in the monom  $\prod_{j \in X_i} m_{j, \sigma(j)}$ , every multiplicand  $m_{j, \sigma(j)} = \pm 1$ , hence each sum

$$\sum_{\sigma \in \mathcal{D}(X_i), j \neq i} \text{sign}(\sigma) \prod_{j \in X_i} m_{j, \sigma(j)}$$

has the same parity as the parity of the number of all derangements in  $\mathcal{D}(X_i)$ ; we shall see in a second that their number  $|\mathcal{D}(X_i)|$  is indeed odd, but let us first complete the proof: if  $\mathcal{D}(X_i)$  is indeed odd, then  $a_1$  is the sum of 101 odd numbers and is therefore odd.

Why the number of derangements of a set of 100 (or any even number) of elements is odd? This is a classical fixed-point free theorem from finite group theory: the inverse of a derangement is a derangement, so the map  $\sigma \mapsto \sigma^{-1}$  is an involutive map of  $\mathcal{D}(X_i)$  onto itself; derangements which are not involutions (that is,  $\sigma \neq \sigma^{-1}$ ) come in pairs  $\{\sigma, \sigma^{-1}\}$ . Therefore the parity of  $\mathcal{D}(X_i)$  is the same as the parity of the set of involutive derangements, that is, permutations of  $X_i$  made of exactly 50 pairwise disjoint cycles  $(j, k)$  of length 2. But their number is odd – for example, because products of 50 pairwise disjoint cycles of length 2 are conjugate in the symmetric group  $\text{Sym}_{100}$  and belong to centers

of Sylow 2-subgroups in  $\text{Sym}_{100}$ ; of course, there are more elementary proofs<sup>†</sup>, but this is the one that instantly comes to my mind.

#### 4.5. *Further generalisations*

First of all, observe that the matrix solution works for an arbitrary field of characteristic 0 – and perhaps for an arbitrary field of characteristic 2.

In odd characteristic, the statement is no longer true: the set of residues modulo 5

$$0, 1, 2, 3, 4$$

is peculiar:

$$1 + 4 \equiv 2 + 3, 0 + 2 \equiv 3 + 4, 0 + 4 \equiv 1 + 3, 0 + 1 \equiv 2 + 4, 0 + 3 \equiv 1 + 2.$$

It is not surprising: in odd characteristic, the concept of parity has no sense.

#### 4.6. *The moral of this tale*

As you had a hance to see see, parity issues continued to chase us to the bitter end. A naive, at the first glance, problem about 101 points has impressive structural depth.

The following well-known image is one of many produced by Anatoly Timofeevich Fomenko in his famous exploration of connections between mathematics and visual arts. It is called “2-adic solenoid”, Figure 1.

I have to admit that I have no vaguest idea why it is 2-adic and why it is a solenoid. Apparently what shown in this paper is just a tip of an iceberg, 2-adicity goes even deeper into the heart of mathematics.

### *Acknowledgements*

I thank Alexander Karabegov, Hovhannes Khudaverdyan, and James Montaldi for their kind permission to use their ideas and proofs.

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<sup>†</sup>A simple recurrent relation can be found, for example, in J. C. Baes, *Let's get deranged!* <http://math.ucr.edu/home/baes/qg-winter2004/derangement.pdf>: if  $!n$  denotes the number of derangements on  $n$  symbols, then

$$!2 = 1 \text{ and } !(n+1) = (n+1) \cdot !n + (-1)^{n+1}.$$

It easily follows by induction that  $!n$  is odd if and only if  $n$  is even.



FIGURE 1. A. T. Fomenko. 2-adic solenoid.

### *About the Author*

I am a research mathematician but I have 40+ years of teaching experience at secondary school and university level in four different countries with four different education systems; since 1998 I am a Professor of Pure Mathematics at the University of Manchester.

I also have an interest in cognitive aspects of mathematical practice; see my book *Mathematics under the Microscope*, [<sup>†</sup>], which explains a mathematician's outlook at psycho-physiological and cognitive issues in mathematics and mathematics education. Some of my papers on mathematics education can be found in my personal online journal/blog *Selected Passages From Correspondence With Friends* [<sup>‡</sup>].

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<sup>†</sup>A. V. Borovik, *Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice*. Amer. Math. Soc., Providence, RI, 2010. 317 pp. ISBN-10: 0-8218-4761-9. ISBN-13: 978-0-8218-4761-9. Available from <http://www.ams.org/bookstore-getitem/item=mbk-71>.

<sup>‡</sup>Selected Passages From Correspondence With Friends. ISSN 2054-7145. <http://www.borovik.net/selecta/>.