Matrix Algebra
Lecture Notes

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1 What is Matrix Algebra?

1.1 Linear forms

It is well-known that the total cost of a purchase of amounts (in kilograms) g_1, g_2, g_3 of some goods (say, apples, bananas, and oranges) at prices p_1, p_2, p_3 liras per kilogram, respectively, is an expression

$$p_1g_1 + p_2g_2 + p_3g_3.$$

Expressions of this kind,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

are called **linear forms in variables** $x_1, \ldots, x_n$ **with coefficients** $a_1, \ldots, a_n$. We shall use an abbreviated notation involving the summation symbol:

$$a_1x_1 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i$$

Matrix Algebra studies the mathematics of linear forms.

1.2 Notation

Over the course, we shall develop increasingly compact notation for operations of Matrix Algebra. In particular, we shall discover that a linear

$$p_1g_1 + p_2g_2 + \cdots + p_ng_n$$

can be very conveniently written as

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

and then abbreviated

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = PG,$$

where

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

an expressions like

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n$ are real numbers, will be called **row vectors**, numbers a_i will be called **components** or **elements** of A. The set of all row vectors with n components will be denoted $\mathbb{R}_n$.

Similarly, we call expressions like

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

column vectors with n components; their set is denoted $\mathbb{R}^n$.

It is useful to treat real numbers as vectors with one component: $\begin{bmatrix} a \end{bmatrix}$. Notice that $\mathbb{R}_n = \mathbb{R}^n$; a vector with single component is simultaneously a row and a column.

1.3 Vector spaces

$\mathbb{R}_n$ and $\mathbb{R}^n$ are vector spaces, that is, vectors of the same kind and with the same number of components can be added componentwise

$$\begin{bmatrix} a_1, & a_2, \cdots, & a_n \end{bmatrix} + \begin{bmatrix} b_1, & b_2, \cdots, & b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1, & a_2 + b_2, \cdots, & a_n + b_n \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \cdots \\ a_n + b_n \end{bmatrix}$$

and multiplied by scalars (that is, numbers)

$$\lambda \left[a_1, \;\; a_2, \cdots, \;\; a_n\right] \;=\; \left[\lambda a_1, \;\; \lambda a_2, \cdots, \;\; \lambda a_n\right]$$

$$\lambda \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \;=\; \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{bmatrix}$$

This operations have the following easy to prove properties (axioms of a **vector space**): for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a vector space V and for all scalars c and d, we have

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There is zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each $\mathbf{u}$ there exists $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
8. $1\mathbf{u} = \mathbf{u}$.

1.4 A brief remark about physics

Physicists use even shorter notation and, instead of

$$p_1g_1 + p_2g_2 + p_3g_3 = \sum_{i=1}^{3} p_i g_i$$

write

$$p_1g^1 + p_2g^2 + p_3g^3 = p_i g^i,$$

omitting the summation sign $\sum$ entirely. This particular trick was invented by Albert Einstein, of all people. **I do not use "physics" tricks in my lectures.**

1.5 Warning

Increasingly compact notation leads to increasingly compact and abstract language.

Unlike, say, Calculus, Matrix Algebra focuses not only on procedures, but also on development of a special mathematics language.

Systems of linear equations

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real numbers. The subscript n can be any natural number.

A **system of simultaneous linear equations** is a collection of one or more linear equations involving the same variables, say $x_1, \ldots, x_n$. For example,

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned}$$

We shall abbreviate the words "**a system of simultaneous linear equations**" just to "**a linear system**".

A **solution** of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true identity when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For example, in the system above $(2, 1)$ is a solution.

The set of all possible solutions is called the **solution set** of the linear system.

Two linear systems are **equivalent** if the have the same solution set.

We shall be use the following **elementary operations** on systems od simultaneous liner equations:

Replacement Replace one equation by the sum of itself and a multiple of another equation.

Interchange Interchange two equations.

Scaling Multiply all terms in a equation by a nonzero constant.

Note: The elementary operations are reversible.

Theorem: Elementary operations preserve equivalence.

If a system of simultaneous linear equations is obtained from another system by elementary operations, then the two systems have the same solution set.

We shall prove later in the course that a system of linear equations has either

- no solution, or
- exactly one solution, or
- infinitely many solutions,

under the assumption that the coefficients and solutions of the systems are *real numbers*.

A system of linear equations is said to be **consistent** it if has solutions (either one or infinitely many), and a system in **inconsistent** if it has no solution.

Solving a linear system

The basic strategy is

> to replace one system with an equivalent system (that is, with the same solution set) which is easier to solve.

Existence and uniqueness questions

- Is the system consistent?
- If a solution exist, is it *unique*?

Equivalence of linear systems

- When are two linear systems equivalent?

Lecture 2 Row reduction and echelon forms [Lay 1.2]

Matrix notation

It is convenient to write coefficients of a linear system in the form of a matrix, a rectangular table. For example, the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 &= 2 \\ x_2 + x_3 &= 3 \end{aligned}$$

has the **matrix of coefficients**

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and the **augmented matrix**

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix};$$

notice how the coefficients are aligned in columns, and how missing coefficients are replaced by 0.

The augmented matrix in the example above has 3 rows and 4 columns; we say that it is a 3×4 matrix. Generally, a matrix with m rows and n columns is called an $\mathbf{m \times n}$ **matrix**.

Elementary row operations

Replacement Replace one row by the sum of itself and a multiple of another row.

Interchange Interchange two rows.

Scaling Multiply all entries in a row by a nonzero constant.

The two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Note:

- The row operations are reversible.
- Row equivalence of matrices is an equivalence relation on the set of matrices.

Theorem: Row Equivalence.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

A **nonzero** row or column of a matrix is a row or column which contains at least one nonzero entry.

We can now formulate a theorem (to be proven later).

Theorem: Equivalence of linear systems.

Two linear systems are equivalent if and only if the augmented matrix of one of them can be obtained from the augmented matrix of another system by frow operations and insertion / deletion of zero rows.