Being in Control[1]

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Abstract

In this paper, I discuss emotions related to a person's control (or lack of control) of his/her mathematics:

sense of danger; sense of security; confidence, feeling of strength; feeling of power;

which eventually lead to the ultimate emotion of mathematics:

realisation that you know and understand something that no-one else in the world knows or understands – and that you ***can prove*** *that.*

These higher level emotions are not frequently discussed in the context of mathematics education – but, remarkably, they are known not only to professional research mathematicians, but also experienced by many children in their first encounters with mathematics.

And I dare to suggest that there is another overarching emotion well known to many professional mathematicians: the feeling of a deep connection with the "inner child". I will focus on a child's perception of mathematics, but will start my narrative from a prominent episode in the history of "adult" mathematics.

Keywords

Mathematics, emotion, control, taming, naming, nomination.

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[1] This is the last pre-publication version of the paper which will appear in U. Xolocotzin (ed,), **Understanding Emotions in Mathematical Thinking and Learning**, Elsevier, 2017 (apparently, as Chapter 3 – all these details, as well as page numbers, ISBN, etc., will be added as the book progresses through the production pipeline – but no editorial changes in the text itself will be reflected in this version). Accepted for publication on 27 Sep 2016.

Naming Infinity

I start my narrative with reminding an important episode in history of mathematics – it was analysed by Graham and Kantor (2009) in their book *Naming Infinity*[2]. It is a fascinating story of Russian mathematicians of the early 20th century who found in spiritual teaching of *Name Worshiping*, an esoteric stream of Russian Orthodox Christianity, the strength to do things that were inaccessible to their Western colleagues; they developed the emerging set theory – a challenging and paradoxical branch of mathematics – further and further towards more and more intricate degrees of infinity because they were not afraid to name infinity; after all, their religion not just allowed – demanded from them to name the God.

Naming Infinity is set not so much in the mathematical context as in the realm of the wider spiritual quests of Egorov, Luzin and their mathematician disciples. Mathematically, Egorov and Luzin followed the great French school of analysis of the time; philosophically, they were guided by Father Pavel Florensky, famous Russian Orthodox theologian, philosopher and polymath. Among other things, Florensky was an electrical engineer and a prominent theoretician of the Russian Symbolism movement in arts. Moscow mathematics was surrounded by a tangled knot of religious, philosophical, ideological, political, aesthetic currents and undercurrents of pre-revolutionary and revolutionary Russia.

There is one aspect of the Name Worshiping / Set Theory conjunction that especially attracts my attention: a touching childishness of Florensky's and Luzin's outlook, their somewhat naive but open and sincere view of the world. This was very much part of *Zeitgeist*:

> *Только детские книги читать,*
> *Только детские думы лелеять ...*
> (О. Мандельштам)[3]

This could be disputed, but one might as well say that Luzin's sincere childishness contributed to his political infantilism that caused so much trouble to him in his later life.

Luzin and his mathematician friends were brave as only children could be brave. Children name the world around them – for them, it is a way of controlling the world. They are happy to use names suggested by adults. But if they have not heard an appropriate name, they are not afraid to invent the names of their own.

As soon as a child encounters mathematics, another, an ideal world starts to grow inside of his or her mind – the world of mathematics, and a child strives to control this world. Here, children also need names – and again, they are not afraid to make new ones.

Quest for control

I systematically collect my mathematician colleagues' testimonies about challenges in their

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[2] A very important review of the book is Sinkevich (2012).

[3] *Only to read childrens' books,*
Only to love childish things ...
(O. Mandelstam, translated by A. S. Kline)

early learning of mathematics (and I now have hundreds of the stories). A common thread in the stories is the same as in *Naming Infinity*: children need to *control* the concepts, objects, and structures they face in mathematics, and they achieve the control by *naming* mathematical entities.

As we shall soon see, controlling ideal objects is a highly emotional affair. My correspondents frequently told me that they suffered more from their inability to communicate their difficulties to adults than from the mathematical difficulties as such – they had no shared language with adults around them.

When I started to collect the childhood stories of mathematics (analysed in detail in my forthcoming book *The Shadows of the Truth* (Borovik 2017), I did not expect *being in control* becoming a recurrent theme in childhood testimonies of my colleagues. I also did not anticipate the depth of emotions involved.

When one asks a question to a mathematician, one has to be prepared to get a whole theory as an answer. One of the first stories came from Leo Harrington [1] and instantly crystallized the 'being in control' theme. But LH's story is best told in his own words:

> *I have three stories that I think of as related. My stories may not be of the kind you want, since they involve no mathematics; but for me they involve some very primitive meta-mathematics, namely: who is in control of the meta-mathematics.*
>
> *This is my mother's memory, not mine. When I was three my mother pushed my baby cart and me to a store. At the time prices were given by little plastic numbers below the item. When we got home my clothes revealed lots of plastic numbers. My mother made me return them.*
>
> *When I was nine in third grade I came home from school with a card full of numbers which I had been told to memorize by tomorrow. I sat on my bed crying; the only time I ever cried over an assignment. The card was the multiplication table.*
>
> *When I was around twelve, in grade school, a magazine article said how many (as a decimal number) babies were born in the world each second. The teacher asked for someone to calculate how many babies were born each year. I volunteered and went to the blackboard. When I had found how many babies were born each day, the number had a .5 at the end. The teacher said to forget the .5. I erased the .5.*

Every teacher and parent knows that children can be very sensitive to issues of control. Not surprisingly, the memories that are described by mathematicians as their first memories of mathematics are frequently memories of attempting to control the world.

Listen to Victor Maltcev [2]:

> *When I was 5, I was once playing with toys early in the morning while all the rest were sleeping. When my mother saw me alone, she thought I would like some company and asked my elder brother to go to me. He came and immediately started messing around soldiers. I shouted at him and asked to put everything were it used to be. He put them back but I said he did not. He asked why but I could not find any explanation for the feeling that the probability of that is 1.*

Or listen to Jakub Gismatullin [3]:

> *The wallpaper near my crib has been put up in a very carelessly way. I remember when I was at around 4 or 5 years old, I got up every morning in a bad mood because of this. Every morning I was trying to move some parts of the wallpaper in my mind. However, after some time (2–3 year) I got used to see inexact pattern. Some time [later] I even imagined the whole plane covered by the wallpaper, however not in a perfect way, but in a way that looks around my crib. That is, I think I have found a certain shape on a wallpaper near my crib, that can be used to tessellate the whole plane (of course this shape contains some inaccuracies).*
>
> *In our flat in Poland we decided not to have a wallpaper, but just a plain wall. My personal reason to make this choice was not to irritate my 2-years old daughter.*

Or read a testimony SC [4]:

> *As a very young lad walking home from school in the Euclidean grid of streets that are the suburbs of Chicago, I thought about avoiding sidewalk cracks: "Step on a crack, break you back." Somehow I knew, or had been told by an older brother that lines were infinite. I reasoned that I didn't know if sidewalk cracks were perfectly regular, and the cracks running north to south from a block to the east might extend to my path. It was Chicago and from the point of view of a child, virtually infinite, so there clearly was no way to avoid sidewalk cracks. I missed an opportunity to become obsessive compulsive.*

Yes, SC missed a chance. In their extreme cases, mathematical obsessions could become clinical – but I omit discussion of medical details (in particular, the role of autism spectrum conditions) from the paper – more details can be found in my forthcoming book *Shadows of the Truth,* Borovik (2017). Here I will only record that Simon Baron-Cohen (famous for his suggestion that mathematicians should avoid marriages between them because of a higher risk of producing autistic offspring) emphasizes that:

> *People with autism not only notice such small details and sometimes can retrieve this information in an exact manner, but they also love to predict and control the world.* (Baron-Cohen (2016), p.139)

Children seek control because, left unsupported, they start to feel dangers. This is a testimony from Pierre Arnoux [5]:

> *When I was 10 years old, I remember very clearly the feeling I had when I first learnt the idea of a variable. The best comparison is that I felt I walked on very thin ice, which could break at any moment, and I only felt safe when I arrived at the solution.*

Olivier Gerard [6]:

> *I concur with Pierre Arnoux with a similar image.*

When I started solving equations in junior high school, I had the feeling that the variable was a very precious thing, that could become ugly if badly handled and following the rules such as "passing numbers from one side to the other and inverting signs" carefully was like walking softly or stepping slowly when moving a large stacks of things or books in one's hands. If something was done too roughly the things ended on the floor, some of them bruised or broken.

Please notice that that awareness of danger described by Pierre Arnoux and Olivier Gerard is not the same as paralysing fear, it is a stimulus for being alert.

The quest for control makes children to seek help from adults; usually they can easily communicate to their parents and teachers their questions about the real world. In case of the ideal world of mathematics, finding the common language is much more difficult, and children feel bitterly disappointed by their inability to get help, and even more frustrated when they see the outright ignorance of adults.

Lawrence Braden [7]:

When I was twelve years old, Mr. P, my math (maths) teacher, told the class that π equalled 22/7. Also that π equalled 3.1416. [...]

Excited, I went home with the grandiose notion of finding π to a hundred decimal places by the process of long division, and wondered if anyone had ever done that sort of thing before. Well, of course, I kept getting the wrong answer! Not 3.1416 at all! I did it over four or five times, and was really disturbed. The book said that π equalled 3.1416 and the teacher said that π equaled 3.1416 so I logically came to the conclusion that I did not know how to do long division! A truly disturbing notion; I thought I was pretty good at it and here I couldn't even do this simple problem!

"Oh", Mr. P said the next day. "I didn't mean that π was exactly equal to 22/7." It was at that point that I learned not to take everything a maths teacher (or any sort of teacher) said as hewn in stone. Years later I came upon Niven's truly beautiful and elementary proof of the irrationality of π, and Lindemann's proof of its transcendence.

The next testimony comes from Nicola Arcozzi [8]: his trully impressive mathematical discovery was completely ignored by her parents.

I was about eight-nine years old (Italian third-fourth grade) and I was learning about continents. "Is Australia a continent or an island?" I asked my father. He answered it was both a continent and an island; an answer I found deeply unsatisfactory. I thought for a while about islands and what makes them different from continents, until – weeks later – I reached the conclusion that, by stretching and contracting, Eurasia could be an island of the Oceans as well as an island of the Como lake ("all its shores are on the Como lake").

A sunny day right after rain I was walking with my mother, I pointed to a puddle and I said: "we are on the island of that puddle".

She shrugged and replied "why do you always say such stupid things". (Only many years *afterwards I learned that was part of something called topology).*

Relations with peers can also be complicated, as explained by RW [9]:

I became unpopular with my class mates and some teachers because sometimes, maths questions appeared ambiguous to me. When a question was hidden in the text and you had to think about how to translate it into a mathematical problem, I frequently came up with at least one solution different from what the teacher expected, referring to different ways of understanding what was written. I used to be very stubborn when a teacher would try to "sell me" that only one unique answer was correct. Of course I was wrong sometimes and just misunderstood what was said, but sometimes I was right and some teachers made me respect them a lot by discussing my views in an open way (independent of whether I was right or wrong).

In later sections we shall discuss children's perception of infinity, and the following almost unbelievable story, from GCS [10], is closely related to this theme.

Age 6, a state primary school in a working class London area. I always enjoyed playing with numbers. A teacher tried to tell us that when you broke a 12 inch ruler into two pieces that were the same, each would be 6 inches long.

I went to see her, because I couldn't see how you could break the ruler into equal pieces, because of the point at 6; it wouldn't know which piece to join.

In adult notation [0,6] *and* [6,12] *are not disjoint, but* [0,6), [6,12] *are not isometric. No matter how I tried to explain the problem, she didn't understand. It was a valuable lesson, because from that point on my expectations of schoolteachers were much reduced.*

If we reconstruct this truly spooky episode in adult terms, we have to admit that a six years old boy was concerned with the nature of *continuum*, one of the most fundamental questions of mathematics.

A philosophically inclined reader will immediately see a parallel with Plato's Allegory of the Cave: children see shadows of the Truth and sometimes find themselves in a psychological trap because their teachers and other adults around them see neither Truth, nor its shadows.

But I wish to make that clear: I include in my book Borovik (2017) only those childhood stories where I can explain, in rigorous mathematical language, what the child had seen in the Cave. I apply the same criterion to this paper.

Caveats and disclaimers

This paper touches emotions so raw that “Caveats and disclaimers” becomes its focal section.

Even an innocent, at a first glance, observation (I put it in the abstract of this paper) that mastering mathematics brings a gift of personal empowerment, the precious

> *realisation that you know and understand something that no-one else in the world knows or understands – and that you* can prove *that.*

is a very emotive statement as it was reminded to me by one of my colleagues who commented on a earlier version of this paper:

> *What a wonderful example of the macho, competitive, arrogant, dog-eat-dog world of pure mathematics!!!!!*

I accept that pure mathematics can be indeed described as a dog-eat-dog world. I write about that in my book *Mathematics under the Microscope*, Borovik (2009), Section 8.9, and plan to say more elsewhere. But this paper is about children. They do not eat anyone – in most cases, they do not even dare to bite the adult's hand that tortures them.

The stories told to me by my fellow mathematicians are so unusual, so unexpected, and occasionally so spooky that caveats and disclaimers are due.

The stories that my correspondents conveyed to me cannot be independently corroborated or authenticated – they are memories that my colleagues have chosen to remember.

The real life material in my research is limited to stories that my fellow research mathematicians have chosen to tell me; they represent tiny but personally significant and highly emotional episodes from their childhood. So far my approach is justified by the warm welcome it found among my mathematician friends, and I am most grateful to them for their support. For some reason (and the reason deserves a study on its own) my colleagues know what I am talking about!

Also, my colleagues' testimonies are consistent with my own memories of my first encounters with mathematics, their joys and sorrows. I do not include my childhood stories in this paper, but some of them can be found in Borovik (2017).

I direct my inquiries to mathematicians for a simple but hard to explain reason: early in my work, I discovered that not only lay people, but also school teachers of mathematics and even many professional researchers in mathematics education, as a rule, cannot clearly retell their childhood stories in *mathematical terms*. It appears that only professional mathematicians / computer scientists / physicists possess an adequate language which allows them to describe in some depth their experiences of learning mathematics.

Other people simply express their emotion without giving any mathematical details. The following statements are taken from a Reddit site[4], they are answers to the question

> *Why do so many children (and adults) hate advanced math? Is it how it's taught, or what is taught?*

As the reader will immediately see, this is all about control. And abuse. And violence. Just read the responses:

> cerebral_monkey: *I can definitely say this is why I hated math as a kid. It was always just "memorize this equation" or "if you see a problem that looks like this, follow these steps exactly," without an explanation of how or why.*

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[4] https://www.reddit.com/r/IAmA/comments/4p4poa/im_helping_parents_and_teachers_who_do_calculus/. Accessed 19 August 2016. I do not know the identity or background of contributors.

meatmeatpotato: *My experience with math was always put X here because the book said to, divide there because that just what you have to do. I never understood the larger picture.*

myintellectisbored: *I had a math teacher that managed to make me hate it. I still am not even sure how he did, but he did. Maybe it was his archaic methods or insisting that the methods I used were wrong.* […] *I think the whole "do the math problem the teacher's way" is part of the problem with math instruction. There are so many different ways to solve a problem and still get to the correct solution. And, a lot of the time, the math techniques in school are the more difficult ways to do a problem.*

lucafishysleep: *There is a general stigma, especially among teenagers (both when I was a teenager, and teenagers I know now) about showing enthusiasm or aptitude for anything. Sticking your head above the parapet and saying: "This is who I am, this is what I can do, this is what I like doing" is 'not cool'.*

Everybodygetslaid69: *Teachers who insisted I couldn't/it was wrong to solve problems in my head were the bane of my school career.*

devolve: *I at 14 realized I could solve for x intuitively without really understanding what I did. Instead of getting help understanding my own process I got reprimanded for doing things too fast and not documenting every step.*

Anthro_Fascist: *I absolutely hate it when the book just told me to just "suck it up and accept this equation as fact".*

[deleted entry] *I specifically remember the feeling of defeat I had when I got a geometry question marked wrong in high school because I had derived the distance formula rather than memorized it and regurgitated it.*

FourOfFiveDentists: *It is the most demoralizing experience ever. It makes you angry, depressed and feel like a loser. My experience is that math teachers are beings of pure evil devoid of any emotion*.

IHartRed: *Put my head down and cried in 5th grade because I just couldn't get division. Teacher thought I was sleeping and walked over and sprayed me with a plant mister. Needless to say I got to lift my head with tears streaming down my face with the entire room staring at me. Never tried again.*

I trust no further comments are needed.

Another point that I wish to clarify: I understand that I encroach onto the sacred grounds of developmental psychology. In contrast with the accepted methodology, I stick to individual case studies. Statistics is always instructive (and, as a mathematically educated person, I hope I have a reasonable grasp of statistics), but I would rather understand the intrinsic logic of individual personal stories. I find an ally in the neurologist Vilayanur Ramachandran (2005) who said about statistical analysis (pp. xi–xii):

There is also a tension in the field of neurology between the 'single case study' approach, the intensive study of just one or two patients with a syndrome, and sifting through a large number of patients and doing a statistical analysis. The criticism is sometimes made that it's easy to be misled by single strange cases, but this is nonsense. Most of the syndromes in neurology that have stood the test of time [...] were initially discovered by a careful study of single case and I don't know of even one that was discovered by averaging results from a large sample.

For that reason, I feel that I have to make the following qualifying remarks:

- I am neither a philosopher nor a psychologist.
- This paper is not about philosophy of mathematics, it is about mathematics.
- This paper is not about psychology of mathematics, it is about mathematics.
- This paper is not about mathematics education, it is about mathematics.

As you will soon see, Antoine de Saint-Exupéry has the same level of authority to me as my mathematics education or psychology of mathematics colleagues. Is anything wrong with that?

Taming mathematical entities

When I told these stories to my wife Anna [11], she instantly responded by telling me how she, aged 9, was using the Russian word приручить, "*to tame*", to describe accommodation of new concepts that she learnt at school: the concept had to become tame, obedient like a well trained dog. Importantly, the word was her secret, she never mentioned it to parents or teachers – I was the first person in her life to whom she revealed it.

Anna was not alone in her invention; here is a story from Yagmur Denizhan [12]:

Although I obviously knew the word before, my real encounter with and comprehension of the concept of "taming" is connected with my reading The Little Prince. As far as I can figure out I must have been nearly 12 years old. Saint-Exupéry offered me a good framework for my potential critiques in face of the world of grown-ups that I was going to enter.

I also must have embraced the concept "taming" so readily that it became part of my inner language. Some years ago a friend of mine told me of a scene from our university years:

One day when he entered the canteen he saw me sitting at a table with notebooks spread in front of me but seemingly doing nothing. He asked me what I was doing and I said (though I do not remember having said it, it sounds very much like me) "I am taming the formulae". (Having heard this story I can recall the feeling. Most probably I must have been studying quantum physics.)

Please notice the appearance in this narrative of Antoine de Saint-Exupéry's book *The Little Prince*, with its famous description of *taming*:

"Come and play with me," proposed the little prince. "I am so unhappy."
"I cannot play with you," the fox said. "I am not tamed."
"Ah! Please excuse me," said the little prince.
But, after some thought, he added:
"What does that mean – 'tame'?"
[...]
"It is an act too often neglected," said the fox. "It means to establish ties."
"'To establish ties'?"
"Just that," said the fox."To me, you are still nothing more than a little boy who is just like a hundred thousand other little boys. And I have no need of you. And you, on your part, have no need of me. To you, I am nothing more than a fox like a hundred thousand other foxes. But if you tame me, then we shall need each other. To me, you will be unique in all the world. To you, I shall be unique in all the world ..."

Nomination

To move beyond taming, I have to give a brief explanation of the role of *nomination*, that is, *naming*, in mathematics. To avoid fearful technicalities, I prefer to do that at a childish level, without stepping outside of elementary school arithmetic – trust me, it already contains the essence of mathematics.

One of the rare books on that particular topic, *Children's Mathematics* by Carruthers and Worthington (2006) documents, using dozens of children's drawings, spontaneous birth of mathematics in pre-school children. The very first picture is taken from *Le Petit Prince*, it is Antoine de Saint-Exupéry's famous drawing of a boa constrictor after swallowing an elephant.

In his picture, little Antoine expressed his understanding of some fragment of the world. The picture worked for him because there were two names attached to the picture: *boa constrictor* and *elephant*. And he famously complained how difficult it was to explain the meaning of the picture to adults:

> *Les grandes personnes ne comprennent jamais rien toutes seules, et c'est fatigant, pour les enfants, de toujours et toujours leur donner des explications.*[5]

Right now Şükrü Yalçinkaya, my research collaborator for many years, and I are writing a hard core mathematics paper, Borovik and Yalçinkaya (2016), where we manipulate with mathematical objects made of some abstract symmetries. We think of them as two-sided mirrors, with labels attached on the opposite sides: one of them says "*point*", another one – "*line*", and these labels serve, respectively, as points and lines of some abstract geometry. It is exactly the same kind of the boa constrictor / elephant kind of thinking – as a mathematician, I do not see the difference – and, I wish to emphasise, this vision is fully shared by my co-author.

I have already said that I systematically collect stories from my mathematician colleagues about challenges in their early learning of mathematics. Besides 'being in control', another common thread in the stories is again the same as in *Naming Infinity*: children need *names* for the concepts, objects, and structures they meet in their first encounters with mathematics.

[5] *Grown-ups never understand anything by themselves, and it is tiresome for children to be always and forever explaining things to them.*

A testimony from Jűrgen Wolfart [13] is quite typical:

Probably I was four years old when my mother still forced me to go to bed after lunch for a while and have a little sleep (children don't need this rest after lunch, but parents need children's sleep). Quite often, I couldn't sleep and made some calculations with small integers to entertain myself, and afterwards I presented the results to my mother. Soon, I did not restrict myself to addition ("und") and invented by myself other arithmetic operations – unfortunately I don't remember which, probably "minus" – but I invented also a name for it, of course not the usual one. I don't remember which name, but I remember that my mother reconstructed from my results what operation I had in mind and told me what I did in official terminology. So I forgot my own words for it, but I had a new toy for the siesta time.

Mathematics is a plethora of names, and even memorizing them all could already be a challenging intellectual task for a child. Not surprisingly, the following observation belongs to a poet; it is taken from *Cahiers* by Paul Valéry:

Vu Estaumier, nommé Directeur de l'Ecole Supérieure des PTT. Me dit que, enfant, à 6 ans il avait appris à compter jusqu'à 6 – en 2 jours. Il comprit alors qu'il y avait 7, et ainsi de suite, et il prit peur qu'il fallût apprendre une infinité de noms. Cet infini l'épouvanta au point de refuser de continuer à apprendre les autres nombres. (Valéry (1974))[6]

Notice that a child was frightened not by infinity of numbers, but by infinity of *names*; he was afraid that the sequence of random words lacking any pattern or logic:

un, deux, trois, quatre, cinq, six, …

would drag on and on for ever. I would agree – it was a scary thought; the poor little child was not told that a two dozen of numerals would suffice, that the rest of the arithmetical universe could be built from a handful of simplest names. He was not reassured in time that mathematics can bring safety back by providing very economic means for a systematic production of the infinity of names. This is evidenced by Roy Stewart Roberts [14]:

At some point [...] *I had discovered that you can continue counting forever, using the usual representation of numbers if one ran out of names.*

As soon as a child discovers that he or she can combine "hundred" and "thousand" to form "one hundred thousand", as a soon as a child gets control over the names for numbers, the counting becomes unstoppable.

A testimony from John R. Shackell [15]:

I would have been three years old, getting towards four. My mother was confined for the birth of my sister and so I was being cared for by an aunt. I don't think she had an easy task.

[6] *Seen Estaumier, appointed Head of the "Superior School for Postal Service and Telecommunications". Told me that, as a child, aged six he had learned how to count up to 6 – in two days. He then understood that there was 7, and so on, and was afraid there would be an infinite amount of names to learn. This infinity horrified him to the extent that he refused to keep learning the other numbers.*

> *I would stand on my head on the sofa and read the page numbers from an encyclopaedia. I was very persistent. The conversation went approximately as follows:*
>
> *– "One thousand three hundred and twenty three, one thousand three hundred and twenty four."*
> *– "John, stop that counting."*
> *– "One thousand three hundred and twenty five, one thousand three hundred and twenty six."*
> *– "Oh John do stop that counting."*
> *– "One thousand three hundred and twenty seven. I wish you were one thousand three hundred and twenty seven."*
> *– "Well you wouldn't be so young yourself!"*

It is worth mentioning that John Shackell is a professor of Symbolic Computing; the work of his life is the book *Symbolic Asymptotics* (Shackell (2004)); in lay terms, these words mean computing (moreover, computing automatically, on a computer) names for certain types of infinity. As we can see, as soon as a child has control over the names for numbers, control over the names for infinity also becomes possible – and can even turn into a professional occupation for life.

Another story comes from Theresia Eisenkölbl [16]:

> *My brother and I had learned (presumably from our parents) how counting goes on and on without an end. We understood the construction but we were left with some doubt that you could really count to high numbers, so we decided to count up to a million by dividing the work and doing it in the obligatory nap time in kindergarten in our heads. After a couple of days, we had to admit that it took too long, so we debated whether it was ok to count in steps of thousands or ten-thousands, now that we had counted to one thousand many times. We ended up being convinced that it is possible to count to a million but slightly unhappy that we could not really do so ourselves.*

Names as spells

Nomination (that is, naming, giving a name to a thing) is an important but underestimated stage in development of a mathematical concept and in learning mathematics. I quote mathematician Semen Kutateladze (2007):

> *Nomination is a principal ingredient of education and transfer of knowledge. Nomination differs from definition. The latter implies the description of something new with the already available notions. Nomination is the calling of something, which is the starting point of any definition. Of course, the frontiers between nomination and definition are misty and indefinite rather than rigid and clear-cut.*

And here is another mathematician talking about this important, but underrated concept:

> *Suppose that you want to teach the 'cat' concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractible claws, a distinctive sonic output, etc.? I'll bet not. You probably show the kid a lot of different cats, saying 'kitty'*

each time, until it gets the idea. (Boas (1981))

And back to Kutateladze (2007):

> *We are rarely aware of the fact that the secondary school arithmetic and geometry are the finest gems of the intellectual legacy of our forefathers. There is no literate who fails to recognize a triangle. However, just a few know an appropriate formal definition.*

This is not just an accident: definitions of many fundamental objects of mathematics in the *Elements* are not definitions in our modern understanding of the word; they are *descriptions*.

For example, Euclid (or a later editor of his *Elements*) defines a straight line as

> *a line that lies evenly with its points.*

It makes sense to interpret this definition as meaning that a line is straight if it collapses in our view field to a point when we hold one end up to our eye.

We have to remember that most basic concepts of elementary mathematics are the result of nomination not supported by a formal definition: number, set, curve, figure, etc.

And we also have to remember that as soon as we start using names, we immediately encounter logical difficulties of varying degree of subtleness – especially if we attempt to give a name to a definite object. I should mention in passing the classical Bertrand Russell's analysis of propositions like

> *'The present King of France is bald'*

and arguments like

> *'The most perfect Being has all perfections; existence is a perfection; therefore the most perfect Being exists'*

(see Russell (1905)). Russell points out that the correct reading of the last phrase should be

> *'There is one and only one entity x which is most perfect; that one has all perfections; existence is a perfection; therefore that one exists.'*

and comments further that

> *As a proof, this fails for want of a proof of the premiss* 'there is one and only one entity x which is most perfect'.

According to Russell, a definite nomination (emphasized, as it frequently happens in English, by the use of the definite article 'the') amounts to assertion of *existence* and u*niqueness* of the nominated object and has to be treated with care.

Perhaps, I would suggest introducing a name for an even more elementary didactic act: *pointing*, like pointing a finger at a thing before naming it.

A teacher dealing with a mathematically perceptive child should point to interesting mathematical objects; if a child is prepared to grasp the object and play with it, a name has to be introduced – and, in most cases, there is no need to rush ahead and introduce formal definitions.

Life of definitions in mathematics is full of intellectual adventures; the size of this article does not allow me to go into any details; I would like to mention only that some definitions eventually become *spells*. My old university friend reminded me that Semen Kutateladze in his lectures on functional analysis frequently used a peculiar phrase:

"By reciting standard spells, we prove that ..."

The words "by reciting standard spells" mean here "by invoking the canonical conceptual framework" (and you may even wish to use the word "sacred" instead of "canonical"). Phrases like this can be used when a definition has overgrown itself and has become a meta-definition, a pointer to the whole new host of names, nominations, definitions.

Some conjectures

Everything in this section is strictly conjectural; I am not making any claims, only a few tentative suggestions triggered by re-reading an old book by Russian historian and anthropologists Porshnev (1974)[7].

I propose that specific structures of human brain responsible for religious feelings are archaic voice communication centres that in the pre-historic times used to process voice signals as absolute commands, like a dog processes a command from its master, without separation of the signifier from the signified. Therefore activation of these centres in a modern human (say, in experiments with electrodes inserted in the brain) results in the subject perceiving the most common words as having "supervalue", as revelations. But so do dogs: the master's command is perceived by a dog as a revelation, one master's word instantly changes a friend into a foe.

I follow Porshnev and propose that a prayer is a communication of a person with his/her supervalue centres in the brain. It is easy to suggest that these supervalue centres, being older than the parts of the brain responsible for consciousness, are better connected with various other archaic parts of the brain, and, first of all, with those responsible for emotions, which explains the undisputed psychological value of a prayer:

В минуту жизни трудную,
Теснится ль в сердце грусть,
Одну молитву чудную
Твержу я наизусть ...
(М. Ю. Лермонтов, 1837)[8]

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[7] Porshnev's approach is parallel to, but very different from that of Jaynes (1990).

[8] *When my life is arduous,*
If sadness freezes blood,
I say one prayer marvelous,
I learned it all by heart ...

(M. Yu. Lermontov, 1837. Translated by Yevgeny Bonver, 2000)

When a definition is used as a spell (or a prayer), it invokes (interiorised earlier, on earlier occasions) mechanisms for feasibility filtering of raw mathematical statements and images produced by subconsciousness. In this phrase, I would even downgrade the word "statement": why not say "utterance". But for a person, the perception of processing of a prayer (usually it is described in subtler words, something like "dissolving in a prayer") and of a ritualized definition could be very similar.

In my book Borovik (2009), pp. 138 – 140, I conjecture that mathematics has a facet systematically ignored in mathematics education – it is a language for communication with subconsciousness. Children, when learning mathematics, need command words for subconscious parts of their mind which actually do their mathematics for them.

I'll write about that in more detail elsewhere.

Children and infinity

Infinity is a name that can be adopted and used by a mathematically perceptive child in a very natural way, the same way as a child absorbs the words of mother tongue; even more, a child may start inventing synonyms, because infinity might stand for something real in his mind's eye.

Here are two stories of the discovery of infinity; as you will see, *naming* is its crucial component; the second crucial component is child's *control* over his mental constructions.

First comes a testimony from DD [17]:

When I was 4 [and 2/3] I first went to nursery school. One day a girl came in with a pencil which had a sort of calculator on the back: a series of five or six wheels with 0–9 on each, allowing simple addition, counting, etc. This was in 1943 in New York. Her calculator absolutely fascinated me, and I kept watching as, when the numbers got larger, there would be all 9s and then a new column on the left would pop up with a 1. I just got a feeling for how the whole system operated and it definitely made me feel really satisfied, though I did not know why or what I would do with this information. Also, no one of my friends seemed the least bit interested: I don't think I explained it very well. That weekend, on the Sunday I got up at just before 6 AM and went into my parents bedroom, quietly, as I was allowed to do, went over to the window and looked out down the empty street, at the far end of which was East River Drive as it was then called, bordering the East River. After a bit I started thinking about those wheels. It seemed to me that more important than the 9s were numbers like 100, 1010, 110, 111 then 1000, 1001, 1010 and so on, and I played out these in longer and longer columns in my head until I was absolutely clear how it worked, and I just knew that what I would now call the place-integer system fitted together in a completely satisfactory way. It was still early so I continued thinking about these numbers, and remembered that we used to argue over whether or not there was a largest number. We would make up peculiar names in these arguments (the boys in the nursery class, that is): so somebody would say that a zillion was largest, and someone else might say, no, a squillion was, and so on, nonsense on nonsense. But if these discussions meant anything, I thought, it should all clear itself up in the column pictures I now had in my mind. I then tried to picture the 0/1 arrangement of the largest number, and was tickled at the thought that if I then cranked everything up by rolling the smallest wheel round and then seeing (in my mind's *eye) the spreading effect it had, I would get an even larger number. Great. Then I got upset: I already had the largest number, according to nursery class arguments. So what was going on. I do not know where it came from, but I suddenly realized that there was no largest number, and I could say exactly why not: just roll on one more, or add 1 (I did know addition quite well by then.) Aha! So I woke up my Dad and excitedly told him that there was no largest number, I could show it, and recited what I had thought out. Poor fellow: it was the overtime season and he worked more than 8 hours a day six days a week – he was not impressed. I won't tell you what he said. Later that day my mother was pleased that her first born son had done something, but I don't believe that either of my parents, or even any of the others in the nursery class, ever really understood the point I was making, and certainly never got intense pleasure from thinking about numbers.*

Although this incident is filtered through my decades of doing mathematical physics, it remains clear in my mind and always has.

Leaving the world of mathematics and mathematicians, we may listen to a story from philosopher MM [18]:

I started thinking about death and wanted to convince myself I would never die, instead of thinking about life after death ... So I started thinking about an infinity in this way: first, I assumed that my entire life was only one dream in one night in another life where I am still the same person but could not fully realize that a full life goes on in each dream (an interesting point about personal identity, I guess). Now, that other life would be finite and have only a finite number of nights. So, I thought further that in each night there must be a finite number of dreams, encapsulating a finite number of lives. This was still short of infinity, so I started thinking that in each of these finitely many dreams of the finitely many nights, I would live a life that would in turn contain finitely many nights, which would contain finitely many dreams, and so on. I was not so sure that I was safe that way (i.e. that I would go on living forever), but I convinced myself that these were enough lives to live, so that even if the process would end, I would still have lived enough, and stopped thinking about it.

A reader of my blog, who signed his comment only as JT, remarked that little MM was safe because of Koenig's Lemma in Set Tehory:

Every infinite finitely branching tree has an infinite path (with no repeated vertices).

Edge of the abyss

I have said before that children may feel the dangers of navigating on unknown mathematical terrain. However, when given security and protection, children prefer the blissful ignorance of dangers of the world; and the world of infinity is dead dangerous. Here is a story from Alexander Olshansky [19]:

In 1955 I was 9 years old. My father, Yuri Nikolaevich Olshansky, a lieutenant colonel-

engineer in Russian Air Force, was transferred to a large air base in Engels. Every Sunday on the sport grounds of the base there were some sport competitions. A relay race of

800 meters + 400 meters + 200 meters + 100 meters

was quite popular; it was called Swedish relay. After two or three races I have come to an obvious conclusion that the team wins which has the strongest runner on the first leg (or on the first two legs) because this runner stays in the race for longer.

But the question that I asked to my father was in the spirit of Zeno's paradoxes: if the race continues the same way,

50 meters + 25 meters + ...,

will it be true that the runners will never reach the end of the 4-th circle (one circle is 400 meters)? (My father was retelling my question to his fellow officers; before World War II, he graduated from the Mathematics Department of Saratov University).

Little Sasha was walking on the edge of an abyss; being an educated mathematician, his father had a false sense of security because perhaps he believed that Zeno's "arrow" paradox (of which Swedish relay is an obvious version) is resolved in elementary calculus by summation of the geometric progression

$$800+400+200+100+50+25+\ldots = 1600 = 4 \times 400.$$

This is true; the runners will indeed reach the end of the 4^{th} circle, and fairly quickly.

But if you think that Zeno's paradox ends here, you are wrong; be prepared to face one of its most vicious forms[9]. Indeed, the real trouble starts after the successful finish of the race:

where is the baton?

Indeed, the whole point of the relay is that each runner passes the baton to the runner on the next leg. After the race is over, each runner can honestly claim that he is no longer in possession of the baton because he passed it to the next runner.

I repeat: can you explain *where is the baton*?

In adult terms, this is a classical conundrum of potential and actual infinity – but, as we see, the problem can be formulated in terms accessible to a child. In the ideal world, a teacher of mathematics should follow the dictum from J. D. Salinger's *Catcher in the Rye* and gently guide a child through dangers of mathematics:

I keep picturing all these little kids playing some game in this big field of rye and all. Thousands of little kids, and nobody's around – nobody big, I mean – except me. And I'm standing on the edge of some crazy cliff. What I have to do, I have to catch everybody if they

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[9] See Silagadze (2014) for a more detailed discussion of various versions of Zeno's paradox.

> *start to go over the cliff – I mean if they're running and they don't look where they're going. I have to come out from somewhere and catch them. That's all I'd do all day. I'd just be the catcher in the rye and all.*

Conclusions

In learning and doing mathematics, a person may experience a range of states of mind and emotions:

> frustration, suspicion, hunch, trust/mistrust, confidence, interest, ardour, chase, "going for the kill", concentration, focus, serenity, inspiration, epiphany, joy, ecstasy,

to name a few. This paper focused on a particular one, the feeling of "being in control" – mostly because it is not frequently discussed in the literature.

By ignoring the issue of control in mathematics, mainstream mathematics education ignores the point, the purpose, and the reason for the existence of mathematics. After all, an apocryphal saying describes mathematics as

> *the language of contracts with Nature which Nature accepts as binding.*

As you can see, it is all about control. In our computer age, I would add to that two more points.

> Mathematics is the language of orders to computers.
>
> Mathematics is also the art of finding precise and reproducible solutions to problems that cannot be solved by, or entrusted to, computers.

Blind trust in computers is the ultimate form of the loss of control. But we should not underestimate the intellectual courage of children: children come to this world to be its masters. We simply have to try not turning them into slaves of computer technology.

We also should not underestimate the power of a child's view of the world. I am not prepared to accept the wisdom of the words:

> *When I was a child, I spake as a child, I understood as a child, I thought as a child: but when I became a man, I put away childish things.* (1 Corinthians 13:11)

Indeed I stand for my former (or inner?) child. In my stance, I find support in words of Michael Gromov [20] (for those not in the know: he is a really famous mathematician):

> *My personal evaluation of myself is that as a child till 8–9, I was intellectually better off than at 14. At 14–15 I became interested in math.*
>
> *It took me about 20 years to regain my 7 year old child perceptiveness.*

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The list of people who helped me in many ways in my work on the project reflected in this paper is becoming almost as long as the paper itself; I refer the reader to *Introduction* of my book *Shadows of the Truth* (Borovik 2017).

But, in addition, I thank Adrien Deloro for help with translations from French, and Peter McBurney, Roman Kossak, Robert Kotiuga, Maria Droujkova, and Galina Sinkevich for useful comments on the previous versions of this paper.

I also list brief biographic data of respondents quoted in this paper:

[1] LH is male, American, professor of mathematical logic in an American university.
[2] VM is male, Ukranian, a PhD student in a British university.
[3] JG is male, has a PhD in mathematics, is a professional research mathematician. At the time of described episode, his family was speaking Tatar, Russian and Polish, but mostly Russian.
[4] SC is male, American, professor of mathematics.
[5] PA is male, French, a professor of mathematics in a French university.
[6] OG is male, French, a mathematical researcher and computer scientist in the private industry.
[7 LB is male, American, a recently-retired math teacher of 40 years experience.
[8] NA is male, Italian, has a PhD in mathematics, teaches at a university.
[9] RW is female, German, professor of mathematics in a German university.
[10] GCS is male, English, lecturer of mathematics in a British university.
[11] AB is female, Russian, for many years taught mathematics at a university.
[12] YD is female, Turkish, professor of electrical and electronics engineering in a leading Turkish university.
[13] JW is a professor of mathematics.
[14] RSR tells about himself: "As an adult I obtained a PhD in mathematics [...], and now am retired if mathematicians ever retire." The episode took place before he went to school.
[15] JRS is male, a professor of mathematics.
[16] TE is female, German, has a PhD in Mathematics (and a Gold Medal of an International Mathematical Olympiad), teaches mathematics at a university. At the time of this episode she was 3 years old, her brother 5 years old.
[17] DD is a mathematical physicist, works in a British university. He preferred not to give his full name.
[18] MM is male, French, a professional philosopher with research interests in philosophy of mathematics. The episode took place at age 7 or 8.
[19] AO holds professorships in mathematics in Moscow and the USA. Some of his famous results in group theory can be described as a subtle and paradoxical interplay of finite and infinite.
[20] MG is male, Russian, Professor at the Institut des Hautes Études Scientifiques, Paris, and the Courant Institute of Mathematical Sciences, New York. Recipient of the Abel Prize (2009).

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