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LOGIC AND INEQUALITIES: A REMEDIAL COURSE BRIDGING GCSE AND UNDERGRADUATE MATHEMATICS

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Abstract

This paper is an informal discussion of a lecture course that, subject to approval, I plan to teach next academic year in the Foundation Studies programme in my university. The course is set at the level bridging GCSE Mathematics as it is taught to students up to the age of 16 in secondary schools in England, and undergraduate mathematics courses for ages 18+. It does not overlap A Level Mathematics and Further Mathematics as they are taught in England for ages 16 to 19. The choice of material could be compared with A level *Decision Mathematics*—but does not duplicate it. However, after some adjusting the course could perhaps make a decent alternative to *Decision Mathematics*.

1. *Introduction*

The proposed course is an update of the course that I taught every Autumn since 2003.

Foundation Studies courses in mathematics are intermediate Year Zero courses offered to students who were conditionally accepted to study at the university (mostly in Engineering and other STEM degree programmes), but who had not studied, or had not got good grades in, A Level Mathematics, or to students from overseas whose school diplomas are not recognised in England. It is a big course, 350 students, about 80 of them are foreigners.

Students in the course come from a variety of socioeconomic, cultural, educational and linguistic backgrounds. Just at a level of basic notation, I have to deal with students who, in their school mathematics, were using two different symbols for multiplication:

$$2 \cdot 3 = 6 \text{ and } 2 \times 3 = 6,$$

and three different symbols for division:

$$6/3 = 2; \quad 6 : 3 = 2; \quad 6 \div 3 = 2.$$

Some countries use decimal point: $\pi = 3.1415\ldots$, while others prefer decimal comma: $\pi = 3{,}1415\ldots$—this list can be easily continued.

Even more obstructive are invisible differences in the logical structure of my students' mother tongues. For example, the connective "or" is strictly exclusive in Chinese: "one or another but not both", while in English "or" is mostly inclusive: "one

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or another or perhaps both". Meanwhile, in mathematics "or" is always inclusive and corresponds to the expression "and/or" of the bureaucratic slang. The principle of *material implication*:

it is true that "false" implies "true"

is unacceptable and even offensive to many students for deeply rooted cultural reasons.

My course serves as an introduction to the language of mathematics (well, in its English dialect), and to mathematical thinking. This explains the choice of material which could be compared with A level *Decision Mathematics*—but does not duplicates it.

Other Foundation Studies courses focus on more technical side of pre-calculus and calculus.

2. *Outline of the course*

The course comprises 20 to 24 lectures of 50 minutes each, twice a week over 10 to 12 weeks, plus one tutorial class a week.

Sets (6 lectures). Sets and their elements, equality of sets. The empty set and its uniqueness. Finite and infinite sets. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Subsets, union, intersection and complement. De Morgan's Laws. Boolean algebra of sets.

Logic (6 to 8 lectures). **Propositional Logic**: statements and connectives. Conjunction, disjunction, negation, conditional, their interpretation in human languages. Truth tables. Material implication. Logical equivalence and tautologies. Boolean Algebra of Propositional Logic.
Predicate Logic: predicates and relations. Universal and existential quantifiers. Some basic logic equivalences of Predicate Logic.
Proof. Proof by contradiction. Proof by induction and computation by recursion.

Inequalities (8 to 10 lectures). Inequalities. Methods of proof for inequalities. Solution of inequalities containing unknown variables. Linear inequalities with one or two variables, systems of linear inequalities with two variables. Graphic representation of the solution sets of inequalities. Some simple problems of linear optimisation in two variables. Quadratic inequalities with one variable.

3. *Backgrounds and justification*

3.1. *A very pragmatic justification.*

The new course contains almost everything necessary for mathematically competent writing and handling macroses for time-dependent EXCEL spreadsheets—and not much else. Time-dependent spreadsheets are daily bread of practical computing in engineering and business, they are the principal mathematical tool of project management.

3.2. *Why Logic?*

For two principal reasons, one of which was already mentioned: cultural accommodation of students from overseas. As I said, the connective "or" is strictly exclusive in Chinese. But this is only the beginning: in Croatian, for example, there are two connectives "and": one *parallel*, to link verbs for actions executed simultaneously, and another *consecutive*.

The second reason is that a reasonable grasp of propositional logic is useful for learning programming languages and design of digital circuits, skills which are necessary for almost all engineering disciplines.

3.3. *Inequalities are crucially important for applications of mathematics*

To give just a few examples,

– Anything which contains the word "estimate" in its name, whether it is in Engineering or in Economics, is based on inequalities.
– Anything which contains the word "approximation" in its name, whether it is in Engineering or in Economics, is based on inequalities.
– Anything which contains the word "optimisation" in its name, whether it is in Engineering or in Economics, is based on inequalities.

In addition,

– Inequalities are mathematical tools for control of errors in measurement and in experimental data, as well as for handling rounding errors in computations.
– Statistics is all about inequalities.
– Perhaps most importantly, the instinctive feel of inequalities makes the basis of quick "back-of-the-envelope" estimates and "guesstimates", the essential part of engineering thinking.

In words of Bertrand Russell,

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in inferring that he is an inexact man. Every careful measurement in science is always given with the prob-*

> *able error ... every observer admits that he is likely wrong, and* ***knows about how much wrong he is likely to be.*** [Emphasis is mine.—AB]

3.4. *Inequalities are badly taught at school.*

Markov's Inequality†, the first but fundamental result of the theory of random variables—and the basis of the entire statistics—is no more than a primary school level observation about inequalities and can be formulated as an arithmetic "word problem" about anglers and fish:

> 50 *anglers caught on average* 4 *fish each. Prove that the number of anglers who caught* 20 *or more fish each is at most* 10*.*

A solution is simple. Assume that there were *more than* 10 anglers who caught *at least* 20 fish each; then these 10 anglers caught together *more than* $20 \times 10 = 200$ fish—a contradiction.

Unfortunately, we cannot expect that all students entering British universities are able to produce this argument. There are two reasons for that:

– This is a proof from contradiction—and this is why basic proofs from contradiction are part of the course.
– The argument requires simultaneous handling of two types of inequalities, "*x is more than y*", denoted $x > y$, and '*x is at least y*", denoted $x \geq y$.

Alas, I many times met students who were asking me questions of that kind:

> *How can we claim that* $3 \geq 2$ *if we already know that* $3 > 2$*?*

This fallacy is a symptom of a dangerous condition—logical deficiency. Handling inequalities demands stronger logical skills than mechanical manipulation of equations.

Moreover, inequalities are frequently more important than equations! For example, besides the equation for a straight line in the plane,

$$ax + by - c = 0,$$

closely related inequalities

$$ax + by < c, \quad ax + by \leq c, \quad ax + by \geq c, \quad ax + by > c$$

are no less important: they describe the way the line cuts the plane in two halves (and therefore have natural applications, say, in computer graphics). To give just one example, here is a simple problem:

> *Answer without sketching graphs: do points* $(1, 3)$ *and* $(-2, 4)$ *lie to the same side off the line*
>
> $$2x + 3y - 1 = 0$$
>
> *or belong to the opposite sides?*

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†If X is any nonnegative random variable and $a > 0$, then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Mathematical logic allows to see connections between inequalities and equations which play an important role in many practical problems.

3.5. *The natural affinity of the theory of inequalities and elementary logic*

Inequalities fit happily into the course which starts with sets and logic not only because they need logic, but also because, in a way of reciprocity, systems of simultaneous inequalities provide accessible material for learning and applying techniques of logic, deduction and proof. Basic Boolean Logic: conjunction, disjunction, negation, comes into play very naturally. A system of two simultaneous inequalities is the *conjunction* of inequalities, the solution set of the system is the *intersection* of the solution sets of individual inequalities.

The inequality $x^2 > 1$ is *equivalent to* the *disjunction* of inequalities $x > 1$ and $x < -1$ and the solution set of $x^2 > 1$ is the *union* of the solution sets of $x > 1$ and $x < -1$:

$$\{\, x \mid x^2 > 1 \,\} = \{\, x \mid x > 1 \,\} \cup \{\, x \mid x < -1 \,\}$$

The *negation* of the inequality $x^2 > 1$ is $x^2 \leq 1$, and the equation $x^2 = 1$ is the *conjunction* of inequalities $x^2 \geq 1$ and $x^2 \leq 1$:

$$(\forall x)(x^2 = 1 \leftrightarrow (x^2 \geq 1 \wedge x^2 \leq 1)).$$

Even more remarkable (and the reason why quadratic inequalities need to be discussed not only for their practical importance, but also as an illustrative material for logic), that the inequality $x > 1$ *implies* the inequality $x^2 > 1$,

$$(\forall x)(x > 1 \rightarrow x^2 > 1),$$

but $x^2 > 1$ *does not imply* $x > 1$,

$$\neg(\forall x)(x^2 > 1 \rightarrow x > 1),$$

or, rewriting this statement in a logically equivalent way,

$$(\exists x)((x^2 > 1) \wedge \neg(x > 1));$$

in plain language, it means

there exists x such that $x^2 > 1$ but $x \leq 1$.

Systems of simultaneous inequalities are *predicates*—*unary*, in case of systems of inequalities in one variable, and *binary*—if we have two variables.

3.6. *Is Logic too hard?*

At an elementary restricted level—no, it is not not. Logical formulae that I gave as examples might appear to be excessively complex. But the logical connectives $\neg$, $\wedge$, $\vee$ are routine operators in computer coding. The *universal quantifier* $\forall$ and the *existential quantifier* $\exists$, if used sparingly, help students to develop sharper reasoning skills. In an example above, the statement

it is not true that $x^2 > 1$ implies $x > 1$

translates into symbolic notation as

$$\neg(\forall x)(x^2 > 1 \to x > 1),$$

and is logically equivalent to

$$(\exists x)((x^2 > 1) \wedge \neg(x > 1)),$$

which, in plain language, means

there is x such that $x^2 > 1$ but $x \leq 1$.

Without prior exposition to basic symbolic logic, many students would have difficulties in understanding that the statement

(A) *it is not true that $x^2 > 1$ implies $x > 1$*

is the same as

(B) *there is x such that $x^2 > 1$ but $x \leq 1$.*

My aim, of course, is to help my students to eventually see that (A) and (B) are the same without resorting to logical symbolism. Graphic representation of inequalities (and therefore some basic set-theoretic thinking) is a useful stepping-stone: both (A) and (B) are equivalent to saying that

the set of solutions of $x^2 > 1$ is not a subset of the set of solutions of $x > 1$.

I will restrict the use of alternating quantifiers to gently introduced single change cases, $\forall\exists$ and $\exists\forall$. Indeed, the $\forall\exists$ combination already has to be handled with great care, it triggers the explosion of infinity:

for **every** *number* **there is a** *bigger number,*

$$(\forall x)(\exists y)(x < y).$$

I will definitely avoid the notorious $\forall\exists\forall$, the perilous stumbling block of the ϵ-δ language of the real analysis.

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Disclaimer

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