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"IMPOSSIBLE EXAMINATION" AND THE NATURE OF "MODELLING"

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I analyse a problem in a first year economics examination at Sheffield University (January 2015) that caused a lively debate in media. It is a useful example of "modelling", a growing fad in mathematics education. This is why it deserves a detailed discussion in the mathematics education context.

I quote from the BBC website[†]:

> **Student fury over 'impossible' economics exam**
>
> Final year economics students at Sheffield University are furious after an exam this week contained questions they found "impossible".
>
> The paper, on the economics of cities, contained compulsory questions on topics they had never been taught, say the students.
>
> More than 90% of those who took the exam have now signed an online petition demanding the university investigate.
>
> The university said all questions were based on topics taught in the course.
>
> But, in a tweet, one candidate complained: "Question three may as well have been in Chinese."
>
> Another asked: "How can they write a paper and include questions on something we haven't been taught, or told to research?" [. . .]
>
> As well as students taking a BSc in economics, the paper was sat by students taking a joint BA honours with other subjects, such as politics.
>
> The joint honours students were particularly badly affected as many lacked the mathematical background of the BSc students.

2010 *Mathematics Subject Classification* 97Mxx, 97M40.

[†] J. Burns, *Student fury over 'impossible' economics exam*, `http://www.bbc.co.uk/news/education-31057005`, 30 January 2015.

This is the problem that caused this controversy.

> Consider a country with many cities and assume there are $N > 0$ people in each city. Output per person is $\sigma N^{0.5}$ and there is a coordination cost per person of γN^2. Assume that $\sigma > 0$ and $\gamma > 0$.

(a) What sort of things does the coordination cost term γN^2 represent? Why does it make sense that the exponent on N is greater than 1? [10 marks]
(b) Draw a graph of per-capita consumption as a function of N and derive the optimal city size N. How does it depend on the parameters of σ and γ? Provide intuition for your answers. [10 marks]
(c) Describe which combinations of σ and γ generate a peasant economy, meaning an economy with no cities (or 1-person cities). Why might the values of the parameters σ and γ have changed over time? What do these changes imply in terms of the optimal city size? [10 marks]

As we can see, this is essentially about mathematical modelling as applied to economics and drawing some conclusions from the model. Mathematics involved (use of differentiation for finding extrema of functions) is somewhere at the school/university boundary.

I will use this examination question as an example of how mathematics modelling should not be taught.

1. Part (a)

(a) What sort of things does the coordination cost term γN^2 represent? Why does it make sense that the exponent on N is greater than 1? [10 marks]

This question is an example of a breach of the fundamental rule of modelling:

> **For modelling a certain phenomenon you should know something about this phenomenon.**

I will use as an example two components of coordination costs: travel within city and water supply.

I will also look at an example which some readers may find controversial. It comes from the economy of war: per capita energy output of a nuclear explosion needed for destruction of a city by a nuclear strike. The sad truth is that mathematical models of supply-chain planning and delivery of goods to customers can as well be used to optimise delivery of bombs to their targets.

1.1. *Travel costs*

Assuming that densities of population in all cities are the same, the diameter of the city is proportional to $N^{\frac{1}{2}}$. A normal person has about two dozen of places he/she has to visit: the workplace, shops, cinema, hairdresser, kid's school, etc., and the number of them does not depend on the size of the city. Therefore per capita costs associated with travel within the city are proportional to the diameter of the city, that is, to $N^{\frac{1}{2}}$.

1.2. *Water supply*

Analysing the cost, per capita, of centralised water supply (with water coming from a single source), we may assume that

- cost w of pumping a tonne of water at distance r is proportional to r (this is expressed by writing $w \sim r$);
- number of people living at distances between r and $r + dr$ from the source is proportional to rdr, hence the total cost for a city of radius R is

$$W \sim \int_0^R r \cdot rdr \sim R^3.$$

But $R \sim N^{\frac{1}{2}}$ hence

$$W \sim \left(N^{\frac{1}{2}}\right)^3 \sim N^{\frac{3}{2}},$$

and *per capita* cost of water supply is

$$\frac{W}{N} \sim \frac{N^{\frac{3}{2}}}{N} \sim N^{\frac{1}{2}},$$

Hence this part of *per capita* "coordination" costs is again proportional to $N^{\frac{1}{2}}$.

Where does N^2 used in the examiner's model come from?

1.3. *Destruction of the city by a nuclear strike*

Perhaps the examiner has in mind a model that involves a corrupt city ruler who sets *per capita* taxes in proportion to N^2 and squanders the money?

Imagine that citizens raised in anger against the tyrant, the ruler fled the city and, in revenge, exploded a nuclear charge at the center of the city. The radius R of the circle on the ground destroyed by a point explosion at some height above the ground is proportional to cubic root of the energy E of the explosion, $R \sim E^{\frac{1}{3}}$. In my naive understanding, this is happening because the density of energy of the explosion, after dissipating and averaging over the hemisphere of radius R and therefore of volume $V \sim R^3$, should exceed some threshold "danger" level; however—and regardless of my opinion—this is an officially accepted crude estimate. So, destruction of the city requires an explosion of energy

$$E \sim R^3 \sim \left(N^{\frac{1}{2}}\right)^3 = N^{\frac{3}{2}},$$

and, taken *per capita*, the energy of the explosion should be

$$\frac{E}{N} \sim \frac{N^{\frac{3}{2}}}{N} = N^{\frac{1}{2}}.$$

And the last but not the least: the last formula can be rewritten as

$$N \sim E^{\frac{2}{3}}$$

or, equivalently,

$$S \sim E^{\frac{2}{3}}$$

where $S \sim R^2$ if the *target damage area*.

This formula explains why multiple independently targetable reentry vehicle (MIRV) intercontinental ballistic missiles (IBM) make the backbone of nuclear deterrent: a WIKIPEADIA explains,[†]

> Several small warheads cause much more target damage area than a single warhead alone. This in turn reduces the number of missiles and launch facilities required for a given destruction level.

So, I repeat my question: where does the estimate $\sim N^2$ used in the examiners' model come from?

Perhaps, they used a more detailed knowledge of urban economy that justifies their choice of parameters, but then the subquestion

> Why does it make sense that the exponent on N is greater than 1?

should on its own cost *much* more than 10 points.

The exponents with which N appears in the formula are critically important parameters of the model, and I will address my further concerns in my analysis of Part (c).

[†] `https://en.wikipedia.org/wiki/Multiple_independently_targetable_reentry_vehicle`.

2. Part (b)

This is a routine mathematical question.

(b) Draw a graph of per-capita consumption as a function of N and derive the optimal city size N. How does it depend on the parameters of σ and γ? Provide intuition for your answers. [10 marks]

Solving (b) requires drawing a graph and then finding the point $N = N_0$ where the function

$$\tau(N) = \sigma N^{\frac{1}{2}} - \gamma N^2$$

reaches its maximum. If differential calculus is supposed to be used, this is a routine calculation and yields

$$N_0 = \left(\frac{\sigma}{4\gamma}\right)^{\frac{2}{3}}.$$

3. Part (c)

This part of the question deserves a closer look.

(c) Describe which combinations of σ and γ generate a peasant economy, meaning an economy with no cities (or 1-person cities). Why might the values of the parameters σ and γ have changed over time? What do these changes imply in terms of the optimal city size? [10 marks]

3.1. *Singularities*

At the first glance, students are invited to find the values of parameters σ and γ when the equation for extremal point,

$$\frac{d}{dN}\left(\sigma N^{\frac{1}{2}} - \gamma N^2\right) = 0,$$

yields solution $N = 1$.

I suspect that the examiner has expected from his/her students an unthinking mechanical application of formulae.

What I am really unhappy that students are being asked to draw some conclusions about point $N = 1$ in a model where the input variable N takes values in the range from 1 to $1,000,000$ and has a singularity at the point $N = 0$.

And this is another fundamental rule of modelling:

if you have a singularity in your model, remember:
- **it is armed and dangerous, and should not be approached.**

On the bright side,

- on very rare and precious occasions the presence and nature of singularities could be a divine revelation.

There are people who confidently manipulate models almost entirely made of singularities; many of them are called theoretical physicists. But they are semi-gods, and I do not dare to enter their realm.

And, by the way, the behaviour of a point extremum could be very unstable: just recall how a drop of mercury runs all over the table: it searches the point where its potential energy is minimal.

3.2. *Chimera?*

As I have said in my analysis of part (a), the exponents for N in the model used cause further concerns.

Let us write output per person as σN^α and coordinating costs per person as γN^β and see how our model is affected by variations in α and β. We differentiate exactly as we did it before:

$$\frac{d}{dN}\left(\sigma N^\alpha - \gamma N^\beta\right) = \alpha\sigma N^{\alpha-1} - \beta\gamma N^{\beta-1} = 0,$$

$$\alpha\sigma N^{\alpha-1} = \beta\gamma N^{\beta-1}$$

$$\frac{\alpha\sigma}{\beta\gamma} = N^{\beta-1-\alpha+1}$$

and finally

$$N_0 = \left(\frac{\alpha\sigma}{\beta\gamma}\right)^{\frac{1}{\beta-\alpha}}.$$

In my eyes, this causes concern: the exponent

$$\frac{1}{\beta-\alpha}$$

leads to a very unstable behaviour of the model for values of β close to α, and difference between α and β affects the outcome much more than the values of $\sigma > \gamma$. For example, imagine that

$$\frac{\sigma}{\gamma} = 2$$

(which is an assumption that does not contradict the common sense) and $\alpha = 0.5$ and $\beta = 0.51$, the optimal size of the city is

$$N_0 \approx 1.96^{100} \approx 10^{29};$$

actually, a calculator gives

$$N_0 \approx 168115259631985096896851914515.47.$$

And this is a *very big number*, which indicates an unstoppable (and possibly exponential) growth of the city population.

I conjecture that the model used in the exam paper is a *chimera*: it attempts to cover both rural and urban economies which live by different laws. But this is an issue for economists, not for me.

4. *Conclusion*

It is difficult to avoid the conclusion that this examination problem was born in ‘teaching to the test” learning culture:

- instead of a realistic model, an artificial one used, perhaps with the aim to make mathematical calculations easier
- in particular, the functions $N^{\frac{1}{2}}$ and N^2 in the model look to be chosen as the most familiar to students power functions.

In my opinion this approach to “modelling” is best avoided.

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