

ELEMENTARY CALCULUS: REAL VS ANALYTIC

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ABSTRACT

I discuss a routine example from A Level mathematics, the meaning of an indefinite integral (antiderivative)

$$\int \frac{dx}{x} = \log |x| + \begin{cases} C^- & \text{if } x < 0 \\ C^+ & \text{if } x > 0 \end{cases}$$

and procedures for evaluating associated definite integrals. Most school textbooks incorrectly state

$$\int \frac{dx}{x} = \log |x| + C,$$

while WOLFRAM ALPHA returns

$$\int \frac{dx}{x} = \log x + C.$$

I use this example to demonstrate the crucial real /complex dichotomy of calculus which is unfortunately ignored not only in high school, but also in the undergraduate mathematics education. I do not think that there is need to explain the trade-off between “real” and “analytic” to mainstream high school students—but I believe that the issue has to be addressed at the level of universities and teacher training.

1. *The two calculi: real versus analytic*

In the real calculus, the indefinite integral $\int \frac{dx}{x}$ is evaluated as

$$\int \frac{dx}{x} = \log |x| + \begin{cases} C^- & \text{if } x < 0 \\ C^+ & \text{if } x > 0 \end{cases} ;$$

please notice the independent choice of integration constants C^- and C^+ on the two sides of the singularity $x = 0$; the frequently used formula

$$\int \frac{dx}{x} = \log |x| + C$$

is, strictly speaking, incorrect.

Issues like that one are covered by the umbrella term “*natural domain of definition of a function.*” At a more elementary level, even without referring to integrals, one can ask, for example, whether the following two formulae define the same function:

$$f(x) = \sqrt{x^2} \text{ and } g(x) = (\sqrt{x})^2.$$

It is not widely recognised that such questions touch a deep conceptual divide.

- Do we deal with non-analytic piecewise real analytic functions or with complex analytic functions?
- When integration is involved, which scheme of integral is used: some version of integrals for functions of real variables or the antiderivative of complex analysis?
- In the latter case, how singularities and branches of analytic functions are accounted for?

2. WOLFRAM ALPHA *and* MATHEMATICA

An extra layer of confusion is added by the widespread use of symbolic calculus packages which do not always explain this basic dichotomy to the users.

In 1990 or 1991 I was first shown the very first, perhaps, version of MATHEMATICA on a Macintosh computer, and it produced a wrong answer to my very first question: I asked it to integrate

$$\int \sqrt{x^2} dx.$$

Alas, it returned $x^2/2 + C$. This was disappointing since the expression $\sqrt{x^2}$ could hardly be called artificial: it pops up, for example, every time scalar products of vectors and their lengths are computed, as well as in computations with complex numbers.

It took almost 10 years for Wolfram Research to fix this bug and make MATHEMATICA to see the difference between analytic and non-analytic functions, at least in simpler situations. This is how it solves it now:

HOW TO ENTER INPUT | RANDOM EXAMPLE

\int dx

Compute Online With *Mathematica*

Traditional Form | Input Form | Output Form

$\int \sqrt{x^2} dx =$

$\frac{x\sqrt{x^2}}{2}$

Alas, this appears to be a local patch it does not address the deeper underlying issue. Indeed, MATHEMATICA gives up as soon as the screw is turned just a bit tighter:

Wolfram *Mathematica*
ONLINE INTEGRATOR
The world's only full-power integration solver

HOW TO ENTER INPUT | RANDOM EXAMPLE

\int dx

Compute Online With *Mathematica*

Traditional Form | Input Form | Output Form

Mathematica could not find a formula for your integral. Most likely this means that no formula exists. [\[More information\]](#)

Time to compute: 0.06 second

But the integral is very easy to evaluate, it is glued piecewise from two canonical integrals:


$$\int \frac{dx}{1+x\sqrt{x^2}} = \begin{cases} \frac{\log(-1-x)-\log(-1+x)}{2} + C_1 & \text{if } x < 0 \\ \tan^{-1}(x) + C_2 & \text{if } x \geq 0 \end{cases}$$

But to put our discussion in a proper context we have to take into account that MATHEMATICA computes in the *complex domain*. To see the implications, let us look at a much simpler example, one of the most basic integrals,

$$\int \frac{dx}{x},$$

as computed by MATHEMATICA:

Indefinite integral:

 Step-by-step solution

$$\int \frac{1}{x} dx = \log(x) + \text{constant}$$

The obvious issue here is that the function $1/x$ is defined for all $x \neq 0$, while $\log x$ is defined, at a high school level of understanding, only for real $x > 0$. Then how can one evaluate

$$\int_{-2}^{-1} \frac{dx}{x}?$$

I am afraid that an unexperienced user of WOLFRAM ALPHA could have difficulties in figuring out that MATHEMATICA is using *complex* logarithm, so that, for real $x < 0$, $\log x$ exists but takes complex values (and that the integration constant can also take *complex* values!), and that, apparently, the definite integral above is evaluated by MATHEMATICA as

$$\int_{-2}^{-1} \frac{dx}{x} = \log(-1) - \log(-2).$$

with the right-hand side miraculously happening to be *real*. However, at a level of sophistication a few notches above high school mathematics, this miracle could be easily explained.

Indeed, what is $\log(-1)$? Well, Euler's famous formula

$$e^{\pi i} = -1$$

tells us that

$$\log(-1) = \pi i.$$

Actually, since

$$e^{2k\pi i} = 1, \quad k \in \mathbb{Z}$$

we get infinitely many values

$$\log(-1) = \pi i + 2k\pi i.$$

The choice of k requires some care, but in many cases it is safe to select the *principal value*

$$\log(-1) = \pi i.$$

We can further expect that

$$\log(-2) = \log(-1 \cdot 2) = \log(-1) + \log 2 = \pi i + \log 2.$$

Therefore

$$\int_{-2}^{-1} \frac{dx}{x} = \log(-1) - \log(-2) = \pi i - (\pi i + \log 2) = -\log 2,$$

as one would expect.[†]

This way of evaluation of a quite a routine definite integral is easy and natural for a mature mathematician, but, in my opinion, could be a bit hard on schoolchildren.

The moral of this simple example is that calculus can be developed as a *complex* (*analytic*) calculus, as implemented in MATHEMATICA (and for good reasons: it is easier for a computer), or as a *real* calculus; the latter is the usual approach of high school and undergraduate mathematics, also for serious reasons, such as the relative ease of visualisation of the real calculus: in its complex counterpart, try to imagine or sketch $\log z$ around $z = 0$!. Mature mathematicians freely switch between the two modes, but it is a hard task for a beginner.

In this form, an antiderivative $\log |x|$ of $1/x$ is *real*, and its difference from a complex antiderivative $\log x$ of the same function is hidden in the integration constants C^\mp because

$$\log x - \log |x| = \pi i$$

is constant for real negative x . However, unlike its complex counterpart, a real antiderivative is no longer analytic (although it is piecewise analytic). This and similar formulae is one of the reasons why, in the real calculus, treating functions involving the (non-analytic!) absolute value function

$$|x| = \sqrt{x^2}$$

becomes a natural problem.

Functions of the real calculus are piecewise analytic. Therefore the concept of a *natural domain of definition* of a function, although important on itself, plays in the real calculus an important additional role: it provides a tool for correctly cutting functions in analytic pieces and integrating, differentiating, or otherwise manipulating them piece-by-piece, rearranging, for example,

$$x\sqrt{x^2} = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}.$$

[†]If we were from the very beginning more confident in handling (multiple!) values of $\log(-1)$ and $\log(-2)$, we could do the same calculation quicker:

$$\log(-1) - \log(-2) = \log\left(\frac{-1}{-2}\right) = \log\left(\frac{1}{2}\right) = -\log 2.$$

But, for didactic reasons, I would not recommend to do that in an A level class.

3. MAPLE

From my experience, MAPLE is more efficient in operating with non-analytic piecewise-analytic functions—but at the expense of additional technical complexity. This could be seen from the following example emailed to me by a colleague:

> Here is what maple does

$$\left[\begin{array}{l} > \text{int}(1/(1+x*\text{sqrt}(x^2)), x); \\ \frac{\arctan(\sqrt{\text{csgn}(x)} x)}{\sqrt{\text{csgn}(x)}} \end{array} \right.$$

> Is that so terrible?

In that case, MAPLE evaluates the integral

$$\int \frac{dx}{1 + x\sqrt{x^2}}$$

correctly, but the answer involves a complex (non-analytic) function $\text{csgn}(x)$ which generalises the real *sign* function $\text{sgn}(x)$ and is defined as

$$\text{csgn}(x) = \begin{cases} 1 & \text{if } 0 < \Re(x) \text{ or } \Re(x) = 0 \text{ and } 0 < \Im(x) \\ -1 & \text{if } \Re(x) < 0 \text{ or } \Re(x) = 0 \text{ and } \Im(x) < 0 \end{cases} ,$$

where $\Re(x)$ and $\Im(x)$ are the imaginary and real parts of x (and, moreover, the value of $\text{csgn}(x)$ is not rigidly defined and can be adjusted, within MAPLE, at the user's will). Luckily, in the case of real x the formula for $\text{csgn}(x)$ is much simpler:

$$\text{csgn}(x) = \begin{cases} 1 & \text{if } 0 < x \\ -1 & \text{if } x < 0 \end{cases} ;$$

and the MAPLE's answer can be interpreted as

$$\int \frac{dx}{1 + x\sqrt{x^2}} = \begin{cases} \frac{\arctan(ix)}{i} + C_1 & \text{if } x < 0 \\ \arctan(x) + C_2 & \text{if } x \geq 0 \end{cases} .$$

In the real calculus, the value of

$$\int \frac{dx}{1 + x\sqrt{x^2}},$$

can be easily found by a piecewise application of two entries from the table of canonical integrals and is best expressed as

$$\begin{aligned} \int \frac{dx}{1 + x\sqrt{x^2}} &= \begin{cases} \int \frac{dx}{1-x^2} & \text{if } x < 0 \\ \int \frac{dx}{1+x^2} & \text{if } x \geq 0 \end{cases} \\ &= \begin{cases} \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| + C_1 & \text{if } x < -1 \\ \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| + C_2 & \text{if } -1 < x < 0 \\ \tan^{-1}(x) + C_3 & \text{if } x \geq 0 \end{cases} . \end{aligned}$$

Meanwhile in the piecewise complex analytic approach the solution is

$$\int \frac{dx}{1+x\sqrt{x^2}} = \begin{cases} \int \frac{dx}{1-x^2} & \text{if } x < 0 \\ \int \frac{dx}{1+x^2} & \text{if } x \geq 0 \end{cases}$$

$$= \begin{cases} \tanh^{-1}(x) + C_1 & \text{if } x < 0 \\ \tan^{-1}(x) + C_2 & \text{if } x \geq 0 \end{cases},$$

which coincides with the MAPLE's answer

$$\int \frac{dx}{1+x\sqrt{x^2}} = \begin{cases} \frac{\arctan(ix)}{i} + C_1 & \text{if } x < 0 \\ \arctan(x) + C_2 & \text{if } x \geq 0 \end{cases}.$$

All three answers are equivalent, as one can easily check.

Summary. It is unfortunate that the crucial real / complex dichotomy of calculus is ignored not only in the high school, but also in the undergraduate mathematics education. I do not think there is need to explain the delicate trade-off between “real” and “analytic” to mainstream school students—but I believe that the issue has to be addressed at the level of universities and teacher training.

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