

NAMING THE NUMBERS

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Introduction

In this paper, I discuss emotions related to a person’s control (of lack of control) of his/her mathematics:

sense of danger; sense of security; confidence, feeling of strength; feeling of power.

These higher level emotions are not frequently discussed in the context of mathematics education—but, remarkably, they are known not only to professional research mathematicians, but also experienced by many children in their first encounters with mathematics.

I will discuss “*naming*” of mathematical entities as one of the means of establishing child’s control of his/her “personal”, interiorised mathematics.

I will focus on a child’s perception of mathematics, but will start my narrative from a prominent episode in the history of “adult” mathematics.

1. *Naming Infinity*

Loren Graham and Jean-Michel Kantor told in their book *Naming Infinity* [9] a fascinating story of Russian mathematicians who found in spiritual teaching of Name Worshiping, an esoteric stream of Russian Orthodox Christianity, the strength to do things that were inaccessible to their Western colleagues; they developed the emerging set theory—a challenging and paradoxical branch of mathematics—further and further towards more and more intricate degrees of infinity because

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they were not afraid to name infinity; after all, their religion not just allowed—demanded from them to name the God.



FIGURE 1. *Portrait of Father Pavel Florensky and his friend, Russian religious philosopher Sergei Bulgakov, made in 1917 by the famous Russian symbolist artist Mikhail Vasilyevich Nesterov. The Tretyakov Gallery, Moscow. Source: Wikipedia Commons. Public domain.*

Naming Infinity set not so much in the mathematical context as in the realm of the wider spiritual quests of Egorov, Luzin and their mathematician disciples. Mathematically, Egorov and Luzin followed the great French school of analysis of the time; philosophically, they were guided by Father Pavel Florensky, famous Russian Orthodox theologian, philosopher and polymath. Among other things, Florensky was an electrical engineer and a prominent theoretician of the Russian Symbolism movement in arts. Moscow mathematics was surrounded by a tangled knot of religious, philosophical, ideological, political, aesthetic currents and undercurrents of pre-revolutionary and revolutionary Russia.

There is one aspect of the Name Worshiping / Set Theory conjunc-



FIGURE 2. Vision to the Youth Bartholomew. Mikhail Vasilyevich Nesterov, 1890. The Tretyakov Gallery, Moscow. Source: Wikipedia Commons. Public domain.

Bartholomew was to become St. Sergius of Radonezh, the most venerated spiritual leader of medieval Russia. This painting marked the inauguration of the symbolist movement. I mention Bartholomew in my paper on dyslexia and phonics [4].

tion that especially attracts my attention: a touching childishness of Florensky's and Luzin's outlook, their somewhat naive but open and sincere view of the world. This was very much part of the *Zeitgeist*:

ТОЛЬКО детские книги читать,
ТОЛЬКО детские думы лелеять . . .
(О. Мандельштам)

This could be disputed, but one might as well say that Luzin's sincere childishness contributed to his political infantilism that caused so much trouble to him in his later life.

Luzin and his mathematician friends were brave as only children could be brave. Children name the world around them—for them, it is a way of controlling the world. They are happy to use names suggested by adults. But if they have not heard an appropriate name, they are not afraid to invent the names of their own.

As soon as a child encounters mathematics, another, an ideal world starts to grow inside of his or her mind—the world of mathematics, and a child strives to control this world. Here, children also need names—and again, they are not afraid to make new ones.

2. *Quest for control*

I systematically collect my mathematician colleagues' testimonies about challenges in their early learning of mathematics (and I now have hundreds of the stories). A common thread in the stories is the same as in *Naming Infinity*: children need to *control* the concepts, objects, and structures they face in mathematics, and they achieve the control by *naming* mathematical entities.

As we shall soon see, controlling ideal objects is a highly emotional affair. My correspondents frequently told me that they suffered more from their inability to communicate their difficulties to adults than from the mathematical difficulties as such—they had no shared language with adults around them.

When I started to collect the childhood stories of mathematics (analysed in detail in my forthcoming book *The Shadows of the Truth* [5]), I did not expect *being in control* becoming a recurrent theme in childhood testimonies of my colleagues. I also did not anticipate the depth of emotions involved.

When one asks a question to a mathematician, one has to be prepared to get a whole theory as an answer. One of the first stories came from Leo Harrington[†] and instantly crystallized the 'being in control' theme. But LH's story is best told in his own words:

I have three stories that I think of as related. My stories may not be of the kind you want, since they involve no mathematics; but for me they involve some very primitive meta-mathematics, namely: *who is in control* of the meta-mathematics.

This is my mother's memory, not mine. When I was three my mother pushed my baby cart and me to a store. At the time prices were given by little plastic numbers below the item. When we got home my clothes revealed lots of plastic numbers. My mother made me return them.

When I was nine in third grade I came home from school with a card full of numbers which I had been told to memorize by tomorrow. I sat on my bed crying; the only time I ever cried over an assignment. The card was the multiplication table.

When I was around twelve, in grade school, a magazine article said how many (as a decimal number) babies were born in the world each second. The teacher asked for someone to calculate how many babies

[†]LH is male, American, professor of mathematical logic in an American university.

were born each year. I volunteered and went to the blackboard. When I had found how many babies were born each day, the number had a .5 at the end. The teacher said to forget the .5. I erased the .5.

Every teacher and parent knows that children can be very sensitive to issues of control. Not surprisingly, the memories that are described by mathematicians as their first memories of mathematics are frequently memories of attempting to control the world.

Listen to Victor Maltcev[†]:

When I was 5, I was once playing with toys early in the morning while all the rest were sleeping. When my mother saw me alone, she thought I would like some company and asked my elder brother to go to me. He came and immediately started messing around soldiers. I shouted at him and asked to put everything where it used to be. He put them back but I said he did *not*. He asked why but I could not find any explanation for the feeling that the probability of that is 1.

Or listen to Jakub Gismatullin[‡]:

The wallpaper near my crib has been put up in a very carelessly way. I remember, when I was at around 4 or 5 years old, I got up every morning in a bad mood because of this. Every morning I was trying to move some parts of the wallpaper in my mind. However, after some time (2–3 year) I got used to see inexact pattern. Some time [later] I even imagined the hole plane covered by the wallpaper, however not in a perfect way, but in a way that looks around my crib. That is, I think I have found a certain shape on a wallpaper near my crib, that can be used to tessellate the whole plane (of course this shape contains some inaccuracies).

In our flat in Poland we decided not to have a wallpaper, but just a plain wall. My personal reason to make this choice was not to irritate my 2-years old daughter.

Or to SC:

As a very young lad walking home from school in the Euclidean grid of streets that are the suburbs of Chicago, I thought about avoiding sidewalk cracks: “Step on a crack, break you back.” Somehow I knew, or had been told by an older brother that lines were infinite. I reasoned that I didn’t know if sidewalk cracks were perfectly regular, and the cracks running north to south from a block to the east might extend to my path. It was Chicago and from the point of view of a child, virtually

[†]VM is male, Ukranian, a PhD student in a British university.

[‡]JG is male, has a PhD in mathematics, is a professional research mathematician. At the time of described episode, his family was speaking Tatar, Russian and Polish, but mostly Russian.

infinite, so there clearly was no way to avoid sidewalk cracks. I missed an opportunity to become obsessive compulsive.

Yes, SC missed a chance. In their extreme cases, mathematical obsessions could become clinical— but I omit discussion of medical details (in particular, the role of autism spectrum conditions) from the paper—more details can be found in my forthcoming book *Shadows of the Truth* [5]. Here I will only record that Simon Baron-Cohen (famous for his suggestion that mathematicians should avoid marriages between them because of a higher risk of producing autistic offspring) emphasizes that:

People with autism not only notice such small details and sometimes can retrieve this information in an exact manner, but they also love to predict and control the world. [1, p. 139]

Children seek control because, left unsupported, they start to feel dangers. This is a testimony from Pierre Arnoux[†]:

When I was 10 years old, I remember very clearly the feeling I had when I first learnt the idea of a variable. The best comparison is that I felt I walked on very thin ice, which could break at any moment, and I only felt safe when I arrived at the solution.

Olivier Gerard[‡]:

I concur with Pierre Arnoux with a similar image.

When I started solving equations in junior high school, I had the feeling that the variable was a very precious thing, that could become ugly if badly handled and following the rules such as “passing numbers from one side to the other and inverting signs” carefully was like walking softly or stepping slowly when moving a large stacks of things or books in one’s hands. If something was done too roughly the things ended on the floor, some of them bruised or broken.

Please notice that that awareness of danger described by Pierre Arnoux and Olivier Gerard is not the same as paralysing fear, it is a stimulus for being alert.

The quest for control makes children to seek help from adults; usu-

[†]PA is male, French, a professor of mathematics in a French university.

[‡]OG tells about himself: “To match the kind of characterization of people you use, I am a French-speaking male of 40 years, educated only in French during the first seventeen years of his life, now a mathematical researcher and computer scientist in the private industry. I have been interested in mathematics for an early age, at least 7 by my own recollections and papers and if I take my parents word, at least 4 years old, asking questions and reasoning about quantities and counts of things.”

ally they can easily communicate to their parents and teachers their questions about the real world. In case of the ideal world of mathematics, finding the common language is much more difficult, and children feel bitterly disappointed by their inability to get help, and even more frustrated when they see the outright ignorance of adults.

3. *The quest for rigour*

*We are usually convinced more easily
by reasons we have found ourselves than
by those which have occurred to others.*

Blaise Pascal, *Pensées* (1670)

What was another surprising discovery of my project is that some children really care about the rigour of arguments.

NP[†]:

Here is one thing which I remember well when playing with a compass as a child: after having drawn a circle with the compass, I started at a particular point P on the circle and—not changing the radius of the compass—tried to create successively points on the original circle by getting to the next point clockwise with the help of the compass by using the previous point as centre. After doing this six times, I always seemed to get back almost to the original point P , but never precisely. At that time, I wasn't sure whether this was based on my drawing not being precise (all my points and lines had some or whether it really was something which was only almost true. I cannot precisely say at what age I encountered this puzzle. I am certain that no foreign language was involved at that time.

MR[‡]:

I was about 12 years old, at secondary school. We had been taught about ruler and compasses constructions, including angle bisections, and had spent time practicing various constructions using the geometrical instruments which I enjoyed. The teacher made a remark in class about it being impossible to trisect an angle. Hmm I thought to myself—it was very easy to bisect an angle, why not trisect? I spent a long time working on ways to trisect an angle with ruler and compasses later at home. But, I remember feeling that I could never be sure whether I had done it and my execution of the instruments was not accurate, hence explaining

[†]NP is male, German, has a PhD in mathematics, teaches in a British university.

[‡]MR is female, English, has a PhD, an university lecturer and a researcher in mathematical education.

the error I detected by measuring with the protractor, or whether I had not done it and the small error measured did indicate the fact of non-trisection.

PS I was delighted to find that angles in origami-geometry can be trisected!

Solomon Garfunkel[†]:

One experience immediately comes to mind. I was in third grade, so likely about 8 years old. My teacher asked the class how far the moon was from the sun. I raced to the board and drew a picture with the earth at the bottom, the moon directly overhead in a circular orbit and the sun directly above both. I reasoned that since the Earth was 93 million miles from the Sun and the moon was 250 thousand miles from the earth, all we had to do was to subtract and voila the Moon was 92.75 million miles from the Sun. The teacher was effusive in her praise, lauding my reasoning ability.

When I got home and began to tell my father about the problem (and the praise) I quickly realized how foolish I had been to oversimplify the model as I did and not take into account the movement of the bodies involved. I also realized that, even assuming circular orbits, I didn't have the technical knowledge to actually solve the problem and that there were many (I couldn't have thought infinite) different solutions. I also realized that my teacher thought I was correct, which led to a healthy skepticism of teachers ever after.

Lawrence Braden[‡]:

When I was twelve years old, Mr. P, my math (maths) teacher, told the class that π equalled $22/7$. Also that π equalled 3.1416. [...]

Excited, I went home with the grandiose notion of finding π to a *hundred* decimal places by the process of long division, and wondered if anyone had ever done that sort of thing before. Well, of course, I kept getting the wrong answer! Not 3.1416 at all! I did it over four or five times, and was really disturbed. The *book* said that π equaled 3.1416 and the *teacher* said that π equaled 3.1416 so I logically came to the conclusion that *I did not know how to do long division*! A truly disturbing notion; I thought I was pretty good at it and here I couldn't even do this simple problem!

"Oh", Mr. P said the next day. "I didn't mean that π was *exactly* equal to $22/7$." It was at that point that I learned not to take everything a maths teacher (or any sort of teacher) said as hewn in stone. Years later I came upon Niven's truly beautiful and elementary proof of the irrationality of π , and Lindemann's proof of its transcendence.

[†]SG is male, American, a well-known expert in mathematical education.

[‡]LB is male, American, a recently-retired math teacher of 40 years experience.

The next testimony comes from Nicola Arcozzi[§]: her trully impressive mathematical discovery was completely ignored by her parents.

I was about eight-nine years old (Italian third-fourth grade) and I was learning about continents. “Is Australia a continent or an island?” I asked my father. He answered it was *both* a continent and an island; an answer I found deeply unsatisfactory. I thought for a while about islands and what makes them different from continents, until—weeks later—I reached the conclusion that, by stretching and contracting, Eurasia could be an island of the Oceans as well as an island of the Como lake (“all its shores are on the Como lake”). A sunny day right after rain I was walking with my mother, I pointed to a puddle and I said: “we are on the island of that puddle”.

She shrugged and replied “why do you always say such stupid things”. (Only many years afterwards I learned that was part of something called topology).

A very sad story from Teresa Patten[†]:

When I was a girl of about 7, being instructed in English (my native language), I remember asking my teacher to explain, “What is two plus one?” She told me the answer is three, and explained that if I have two oranges and my friend gives me one orange, I will then have three oranges. “Yes,” I said, “but what *is* it?”

What I was really trying to ask was “What is the nature of number?” I wanted to know how this abstract concept can apply universally to any unit we determine to be a unit, and how this correlates to our sensory experience of things as individual items. In particular, I wanted to know which is more ‘real’, the abstract concept or the thing itself? But of course as a 7-year-old I did not have the mastery of language to express this, and even if I could have done so I sincerely doubt my teacher would have understood the question. So instead of pursuing the idea she concluded that I could not do simple addition and put me into the lowest math group (this was California in the 1970s, and at that place and time children were taught in groups determined by aptitude), which is where I remained until I was about 11.

I think the saddest part of this is that until my late teens I believed I had no ability to do math whatsoever. I simply assumed that my classmates all understood the nature of number and other theoretical questions that seemed so difficult to me. It never occurred to me that the other children probably never even thought to ask them.

[§]NA is male, Italian, has a PhD in mathematics, teaches at an university.

[†]TP is female, has an undergraduate degree in mathematics.

Relations with peers can also be complicated, as explained by RW[‡]:

I became unpopular with my class mates and some teachers because sometimes, maths questions appeared ambiguous to me. When a question was hidden in the text and you had to think about how to translate it into a mathematical problem, I frequently came up with at least one solution different from what the teacher expected, referring to different ways of understanding what was written. I used to be very stubborn when a teacher would try to “sell me” that only one unique answer was correct. Of course I was wrong sometimes and just misunderstood what was said, but sometimes I was right and some teachers made me respect them a lot by discussing my views in an open way (independent of whether I was right or wrong).

In later sections we shall discuss children’s perception of infinity, and the following almost unbelievable story, from GCS[†], is closely related to this theme.

Age 6, a state primary school in a working class London area. I always enjoyed playing with numbers. A teacher tried to tell us that when you broke a 12 inch ruler into two pieces that were the same, each would be 6 inches long.

I went to see her, because I couldn’t see how you could break the ruler into equal pieces, because of the point at 6; it wouldn’t know which piece to join.

In adult notation $[0, 6]$ and $[6, 12]$ are not disjoint, but $[0, 6)$, $[6, 12]$ are not isometric. No matter how I tried to explain the problem, she didn’t understand. It was a valuable lesson, because from that point on my expectations of schoolteachers were much reduced.

If we reconstruct this truly spooky episode in adult terms, we have to admit that a six years old boy was concerned with the nature of *continuum*, one of the most fundamental questions of mathematics.

A philosophically inclined reader will immediately see a parallel with Plato’s Allegory of the Cave: children see shadows of the Truth and sometimes find themselves in a psychological trap because their teachers and other adults around them see neither Truth, nor its shadows.

But I wish to make that clear: I include in my forthcoming book [5] only those childhood stories where I can explain, in rigorous mathe-

[‡]RW is female, German, a professor of mathematics in a German university.

[†]GCS is male, English, a lecturer of mathematics in a British university.

mathematical language, what the child had seen in the Cave. I apply the same criterion to this paper.

4. *Caveats and disclaimers*

The stories told to me by my fellow mathematicians are so unusual, so unexpected, and occasionally so spooky that some caveats and disclaimers are due.

The stories that my correspondents conveyed to me cannot be independently corroborated or authenticated—they are memories that my colleagues have chosen to remember.

The real life material in my research is limited to stories that my fellow research mathematicians have chosen to tell me; they represent tiny but personally significant and highly emotional episodes from their childhood. So far my approach is justified by the warm welcome it found among my mathematician friends, and I am most grateful to them for their support. For some reason (and the reason deserves a study on its own) my colleagues know what I am talking about!

Also, my colleagues' testimonies are consistent with my own memories of my first encounters with mathematics, their joys and sorrows. I do not include my childhood stories in this paper, but they can be found in [5].

I direct my inquiries to mathematicians for a simple but hard to explain reason: early in my work, I discovered that not only lay people, but also school teachers of mathematics and even many professional researchers in mathematics education, as a rule, cannot clearly retell their childhood stories. It appears that only professional mathematicians / computer scientists / physicists possess an adequate language which allows them to describe in some depth their experiences of learning mathematics.

Another point that I wish to clarify: I understand that I encroach onto the sacred grounds of developmental psychology. In contrast with the accepted methodology, I stick to individual case studies. Statistics is always instructive (and, as a mathematically educated person, I claim that I have a reasonable grasp of statistics), but I would rather understand the intrinsic logic of individual personal stories. I find an ally in the neurologist Vilayanur Ramachandran who said about statistical analysis [13, pp. xi–xii]:

There is also a tension in the field of neurology between the ‘single case study’ approach, the intensive study of just one or two patients with a syndrome, and sifting through a large number of patients and doing a statistical analysis. The criticism is sometimes made that it’s easy to be misled by single strange cases, but this is nonsense. Most of the syndromes in neurology that have stood the test of time [...] were initially discovered by a careful study of single case and I don’t know of even one that was discovered by averaging results from a large sample.

For that reason, I feel that I have to make the following qualifying remarks:

- I am neither a philosopher nor a psychologist.
- This paper is not about philosophy of mathematics, it is about mathematics.
- This paper is not about psychology of mathematics, it is about mathematics.
- This paper is not about mathematics education, it is about mathematics.

As you will soon see, Antoine de Saint-Exupéry has the same level of authority to me as my mathematics education or psychology of mathematics colleagues. Is anything wrong with that?

5. *Taming mathematical concepts and objects*

When I told these stories to my wife Anna[†], she instantly responded by telling me how she, aged 9, was using the Russian word *приучить*, “to tame”, to describe accommodation of new concepts that she learnt at school: the concept had to become tame, obedient like a well trained dog. Importantly, the word was her secret, she never mentioned it to parents or teachers—I was the first person in her life to whom she revealed it.

Anna was not alone in her invention; here is a story from Yağmur Denizhan[‡]:

Although I obviously knew the word before, my real encounter with and comprehension of the concept of “taming” is connected with my reading *The Little Prince*. As far as I can figure out I must have been nearly 12

[†]AB is female, Russian, for many years taught mathematics at an university.

[‡]YD is female, Turkish, a professor of computer science in the leading Turkish university.

years old. Saint-Exupéry offered me a good framework for my potential critiques in face of the world of grown-ups that I was going to enter.

I also must have embraced the concept “taming” so readily that it became part of my inner language. Some years ago a friend of mine told me of a scene from our university years:

One day when he entered the canteen he saw me sitting at a table with notebooks spread in front of me but seemingly doing nothing. He asked me what I was doing and I said (though I do not remember having said it, it sounds very much like me) “I am taming the formulae”. (Having heard this story I can recall the feeling. Most probably I must have been studying quantum physics.)

Please notice the appearance in this narrative of Antoine de Saint-Exupéry’s book *The Little Prince*, with its famous description of *taming*:

“Come and play with me,” proposed the little prince. “I am so unhappy.”

“I cannot play with you,” the fox said. “I am not tamed.”

“Ah! Please excuse me,” said the little prince.

But, after some thought, he added:

“What does that mean—‘tame’?”

[...]

“It is an act too often neglected,” said the fox. “It means to establish ties.”

“ ‘To establish ties’?”

“Just that,” said the fox. “To me, you are still nothing more than a little boy who is just like a hundred thousand other little boys. And I have no need of you. And you, on your part, have no need of me. To you, I am nothing more than a fox like a hundred thousand other foxes. But if you tame me, then we shall need each other. To me, you will be unique in all the world. To you, I shall be unique in all the world ...”

6. *Nomination*

Hand-in-hand with taming goes *naming*. I have to give a brief explanation of the role of *nomination*, that is, *naming*, in mathematics. To avoid fearful technicalities, I prefer to do that at a childish level, without stepping outside of elementary school arithmetic—trust me, it already contains the essence of mathematics [3].

One of the rare books on that particular topic, *Children’s Mathematics* by Carruthers and Worthington [8] documents, using dozens of children’s drawings, spontaneous birth of mathematics in pre-school



FIGURE 3. *A boa constrictor after swallowing an elephant, by Antoine de Saint-Exupéry.*

children. The very first picture is taken from *Le Petit Prince*, it is Antoine de Saint-Exupéry’s famous drawing of a boa constrictor after swallowing an elephant, Figure 3.

In his picture, little Antoine expressed his understanding of some fragment of the world. The picture worked for him because there were two names attached to the picture: *boa constrictor* and *elephant*. And he famously complained how difficult it was to explain the meaning of the picture to adults:

Les grandes personnes ne comprennent jamais rien toutes seules, et c’est fatigant, pour les enfants, de toujours et toujours leur donner des explications.

Right now Şükrü Yalçınkaya, my research collaborator for many years, and I are writing a hard core mathematics paper [6], where we manipulate with mathematical objects made of some abstract symmetries (think of them as two-sided mirrors, with labels attached on the opposite sides: one of the says “point”, another one—“line”, and these objects serve, respectively, as points and lines of some abstract geometry. It is exactly the same kind of the boa constrictor / elephant kind of thinking—as a mathematician, I do not see the difference—and, I wish to emphasise, this vision is fully shared by my co-author.

I have already said that I systematically collect stories from my mathematician colleagues about challenges in their early learning of mathematics. Besides ‘being in control’, another common thread in the stories is again the same as in *Naming Infinity*: children need *names* for the concepts, objects, and structures they meet in their first encounters with mathematics.

A testimony from Jürgen Wolfart[†] is quite typical:

Probably I was four years old when my mother still forced me to go to bed after lunch for a while and have a little sleep (children don’t need this rest after lunch, but parents need children’s sleep). Quite often, I

[†]JW is a professor of mathematics.

couldn't sleep and made some calculations with small integers to entertain myself, and afterwards I presented the results to my mother. Soon, I did not restrict myself to addition (“und”) and invented by myself other arithmetic operations—unfortunately I don't remember which, probably “minus”—but I invented also a name for it, of course not the usual one. I don't remember which name, but I remember that my mother reconstructed from my results what operation I had in mind and told me what I did in official terminology. So I forgot my own words for it, but I had a new toy for the siesta time.

Mathematics is a plethora of names, and even memorizing them all could already be a challenging intellectual task for a child. Not surprisingly, the following observation belongs to a poet; it is taken from *Cahiers* by Paul Valéry:

Vu Estaumier, nommé Directeur de l'Ecole Supérieure des PTT. Me dit que, enfant, à 6 ans il avait appris compter jusqu'à 6 – en 2 jours. Il comprit alors qu'il y avait 7, et ainsi de suite, et il prit peur qu'il fallût apprendre une infinité de noms. Cet infini l'épouvanta au point de refuser de continuer apprendre les autres nombres. [17, Tôme II, p. 798]

Notice that a child was frightened not by infinity of numbers, but by infinity of *names*; he was afraid that the sequence of random words lacking any pattern or logic:

un, deux, trois, quatre, cinq, six, ...

would drag on and on for ever. I would agree—it was a scary thought; the poor little child was not told that a two dozen of numerals would suffice, that the rest of the arithmetic universe could be built from a handful of simplest names. He was not reassured in time that mathematics can bring safety back by providing very economic means for a systematic production of the infinity of names. This is evidenced by Roy Stewart Roberts[†]:

At some point [...] I had discovered that you can continue counting forever, using the usual representation of numbers if one ran out of names.

As soon as a child discovers that he or she can combine “hundred” and “thousand” to form “one hundred thousand”, as soon as a child gets control over the names for numbers, the counting becomes unstoppable.

[†]RSR tells about himself: “As an adult I obtained a PhD in mathematics [...], and now am retired if mathematicians ever retire.” The episode took place before he went to school.

Here is a testimony from John R. Shackell[‡]:

I would have been three years old, getting towards four. My mother was confined for the birth of my sister and so I was being cared for by an aunt. I don't think she had an easy task.

I would stand on my head on the sofa and read the page numbers from an encyclopedia. I was very persistent. The conversation went approximately as follows:

- “One thousand three hundred and twenty three, one thousand three hundred and twenty four.”
- “John, stop that counting.”
- “One thousand three hundred and twenty five, one thousand three hundred and twenty six.”
- “Oh John *do* stop that counting.”
- “One thousand three hundred and twenty seven. I wish *you* were one thousand three hundred and twenty seven.”
- “Well you wouldn't be so young yourself!”

It is worth mentioning that John Shackell is professor of Symbolic Computing; the work of his life is the book *Symbolic Asymptotics* [15]; in lay terms, these words mean computing (moreover, computing automatically, on a computer) names for certain types of infinity. As we can see, as soon as a child has control over the names for numbers, control over the names for infinity also becomes possible—and can even turn into a professional occupation for life.

Another story comes from Theresia Eisenkölbl[†]:

My brother and I had learned (presumably from our parents) how counting goes on and on without an end. We understood the construction but we were left with some doubt that you could *really* count to high numbers, so we decided to count up to a million by dividing the work and doing it in the obligatory nap time in kindergarten in our heads. After a couple of days, we had to admit that it took too long, so we debated whether it was ok to count in steps of thousands or ten-thousands, now that we had counted to one thousand many times. We ended up being convinced that it is possible to count to a million but slightly unhappy that we could not *really* do so ourselves.

[‡]JRS is a professor of mathematics.

[†]TE has a PhD in Mathematics (and a Gold Medal of an International Mathematical Olympiad), teaches mathematics at an university. At the time of this episode she was 3 years old, her brother 5 years old.

7. *Names as spells*

Nomination (that is, naming, giving a name to a thing) is an important but underestimated stage in development of a mathematical concept and in learning mathematics. I quote mathematician Semen Kutateladze[‡] [10]:

Nomination is a principal ingredient of education and transfer of knowledge. Nomination differs from definition. The latter implies the description of something new with the already available notions. Nomination is the calling of something, which is the starting point of any definition. Of course, the frontiers between nomination and definition are misty and indefinite rather than rigid and clear-cut.

And here is another mathematician talking about this important, but underrated concept:

Suppose that you want to teach the ‘cat’ concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractible claws, a distinctive sonic output, etc.? I’ll bet not. You probably show the kid a lot of different cats, saying ‘kitty’ each time, until it gets the idea. (Ralph P. Boas, Jr. [2].)

And back to Semen Kutateladze:

We are rarely aware of the fact that the secondary school arithmetic and geometry are the finest gems of the intellectual legacy of our forefathers. There is no literate who fails to recognize a triangle. However, just a few know an appropriate formal definition.

This is not just an accident: definitions of many fundamental objects of mathematics in the *Elements* are not definitions in our modern understanding of the word; they are *descriptions*.

For example, Euclid (or a later editor of his *Elements*) defines a straight line as

a line that lies evenly with its points.

It makes sense to interpret this definition as meaning that a line is

[‡]I do not know whether it is coincidence or not, but Semen Kutateladze is also an expert on history of Soviet mathematics, and, in particular, on “the Luzin affair” [11].

straight if it collapses in our view field to a point when we hold one end up to our eye.

We have to remember that most basic concepts of elementary mathematics are the result of nomination not supported by a formal definition: number, set, curve, figure, etc.

And we also have to remember that as soon as we start using names, we immediately encounter logical difficulties of varying degree of subtleness — especially if we attempt to give a name to a definite object. I should mention in passing the classical Bertrand Russell's analysis of propositions like

'The present King of France is bald'

and arguments like

'The most perfect Being has all perfections; existence is a perfection; therefore the most perfect Being exists'

(see [14]). Russell points out that the correct reading of the last phrase should be

'There is one and only one entity x which is most perfect; that one has all perfections; existence is a perfection; therefore that one exists.'

and comments further that

As a proof, this fails for want of a proof of the premiss *'there is one and only one entity x which is most perfect'*.

According to Russell, a definite nomination (emphasized, as it frequently happens in English, by the use of the definite article 'the') amounts to assertion of *existence* and *uniqueness* of the nominated object and has to be treated with care.

Perhaps, I would suggest introducing a name for an even more elementary didactic act: *pointing*, like pointing a finger at a thing before naming it.

A teacher dealing with a mathematically perceptive child should point to interesting mathematical objects; if a child is prepared to grasp the object and play with it, a name has to be introduced—and, in most cases, there is no need to rush ahead and introduce formal definitions.

Life of definitions in mathematic is full of intellectual adventures;

the size of this article does not allow to go into any details; I would like to mention only that some definitions eventually become *spells*. My old university friend[†] reminded me that Semen Kutateladze in his lectures on functional analysis frequently used a peculiar phrase:

“By reciting standard spells, we prove that . . .”

“By reciting standard spells” means here “by invoking the canonical conceptual framework” (and you may even wish to use the word “sacred” instead of “canonical”). Phrases like this can be used when a definition has overgrown itself and has become a meta-definition, a pointer to the whole new host of names, nominations, definitions.

8. *Some blatant conjectures*

I propose that specific structures of human brain responsible for religious feelings are ancient archaic voice communication centers that in the pre-historic times used to process voice signals as absolute commands, like a dog processes a command from its master, without separation of the signifier from the signified. Therefore activation of these centers in a modern human (say, in experiments with electrodes inserted in the brain) results in the subject perceiving the most common words as having “supervalue”, as revelations. But so do dogs: the master’s command is perceived by a dog as a revelation, one master’s word instantly changes a friend into a foe.

A prayer is a communication of a person with his/her supervalue centers in the brain. It is easy to suggest that these supervalue centers, being older than the parts of the brain responsible for consciousness, are better connected with various other archaic parts of the brain, and, first of all, with those responsible for emotions, which explains the undisputed psychological value of a prayer:

В минуту жизни трудную,
Теснится ль в сердце грусть,
Одну молитву чудную
Твержу я наизусть . . .
(М. Ю. Лермонтов)

When a definition is used as a spell (or a prayer), it invokes (interiorized earlier, on earlier occasions) mechanisms for feasibility filtering of raw mathematical statements and images produced by subconscious-

[†]She prefers to be known only as Owl (*Otus Persapient*).



FIGURE 4. *Vespers* by John Singer Sargent, 1909. I love this painting for the expression of a specific concentration of mind which could be read as mystical—or mathematical, if one is inclined so. What is the difference, after all? This is a common facial expression of a man in communication with the deepest levels of his subconsciousness. Sargent painted *Vespers* in an Orthodox monastery on Corfu, but I have seen the same faces, in the same kind of Mediterranean light, say, at the Centre International de Rencontres Mathématiques (CIRM) in Luminy.

The painting is in Walker Art Gallery in Liverpool, I can attest that the original is much subtler than any reproduction.

Source: WIKIMEDIA COMMONS. Public domain.

ness. In this phrase, I would even downgrade the word “statement”: why not say “utterance”. But for a person, the perception of processing of a prayer (usually it is described in subtler words, something like “dissolving in a prayer”) and of a ritualized definition could be very similar.

9. *Children and infinity*

Infinity is a name that can be adopted and used by a mathematically perceptive child in a very natural way, the same way as a child absorbs the words of mother tongue; even more, a child may start inventing synonyms, because infinity might stand for something real in his mind’s eye.

Here are two stories of the discovery of infinity; as you will see, *nam-*

ing is its crucial component; the second crucial component is child's *control* over his mental constructions.

First comes a testimony from DD[†]:

When I was $4\frac{2}{3}$ I first went to nursery school. One day a girl came in with a pencil which had a sort of calculator on the back: a series of five or six wheels with 0–9 on each, allowing simple addition, counting, etc. This was in 1943 in New York. Her calculator absolutely fascinated me, and I kept watching as, when the numbers got larger, there would be all 9s and then a new column on the left would pop up with a 1. I just got a feeling for how the whole system operated and it definitely made me feel really satisfied, though I did not know why or what I would do with this information. Also, no one of my friends seemed the least bit interested: I don't think I explained it very well. That weekend, on the Sunday I got up at just before 6 AM and went into my parents bedroom, quietly, as I was allowed to do, went over to the window and looked out down the empty street, at the far end of which was East River Drive as it was then called, bordering the East River. After I bit I started thinking about those wheels. It seemed to me that more important than the 9s were numbers like 100, 1010, 110, 111 then 1000, 1001, 1010 and so on, and I played out these in longer and longer columns in my head until I was absolutely clear how it worked, and I just knew that what I would now call the place-integer system fitted together in a completely satisfactory way. It was still early so I continued thinking about these numbers, and remembered that we used to argue over whether or not there was a largest number. We would make up peculiar names in these arguments (the boys in the nursery class, that is): so somebody would say that a *zillion* was largest, and someone else might say, no, a *squillion* was, and so on, nonsense on nonsense. But if these discussions meant anything, I thought, it should all clear itself up in the column pictures I now had in my mind. I then tried to picture the 0/1 arrangement of the largest number, and was tickled at the thought that if I then cranked everything up by rolling the smallest wheel round and then seeing (in my mind's eye) the spreading effect it had, I would get an even larger number. Great. Then I got upset: I already had the largest number, according to nursery class arguments. So what was going on. I do not know where it came from, but I suddenly realized that there was no largest number, and I could say exactly why not: just roll on one more, or add 1 (I did know addition quite well by then.) Aha! So I woke up my Dad and excitedly told him that there was no largest number, I could show it, and recited what I had thought out. Poor fellow: it was the overtime season and he worked more than 8 hours a day six days a week—he was not impressed. I won't tell you what he said. Later that day my mother was pleased that her first born son had done something,

[†]DD is a mathematical physicist, works in a British university. He preferred not to give his full name.

but I don't believe that either of my parents, or even any of the others in the nursery class, ever really understood the point I was making, and certainly never got intense pleasure from thinking about numbers.

Although this incident is filtered through my decades of doing mathematical physics, it remains clear in my mind and always has.

Leaving the world of mathematics and mathematicians, we may listen to a story from philosopher MM[†]:

I started thinking about death and wanted to convince myself I would never die, instead of thinking about life after death ... So I started thinking about an infinity in this way: first, I assumed that my entire life was only one dream in one night in another life where I am still the same person but could not fully realize that a full life goes on in each dream (an interesting point about personal identity, I guess). Now, that other life would be finite and have only a finite number of nights. So, I thought further that in each night there must be a finite number of dreams, encapsulating a finite number of lives. This was still short of infinity, so I started thinking that in each of these finitely many dreams of the finitely many nights, I would live a life that would in turn contain finitely many nights, which would contain finitely many dreams, and so on. I was not so sure that I was safe that way (i.e. that I would go on living forever), but I convinced myself that these were enough lives to live, so that even if the process would end, I would still have lived enough, and stopped thinking about it.

A reader of my blog, who signed his comment only as JT, remarked that little MM was safe because of König's Lemma:

Every infinite finitely branching tree has an infinite path (with no repeated vertices).

I have said before that children may feel the dangers of navigating on unknown mathematical terrain. However, when given security and protection, children prefer the blissful ignorance of dangers of the world; and the world of infinity is dead dangerous. Here is a story from Alexander Olshansky[‡]:

In 1955 I was 9 years old. My father, Yuri Nikolaevich Olshansky, a lieutenant colonel-engineer in Russian Air Force, was transferred to a large air base in Engels. Every Sunday on the sport grounds of the base there

[†]MM is male, French, a professional philosopher with research interests in philosophy of mathematics. The episode took place at age 7 or 8.

[‡]AO holds professorships in mathematics in Moscow and the USA. Some of his famous results in group theory can be described as a subtle and paradoxical interplay of finite and infinite.

were some sport competitions. A relay race of

$$800 \text{ meters} + 400 \text{ meters} + 200 \text{ meters} + 100 \text{ meters}$$

was quite popular; it was called Swedish relay. After two or three races I have come to an obvious conclusion that the team wins which has the strongest runner on the first leg (or on the first two legs) because this runner stays in the race for longer.

But the question that I asked to my father was in the spirit of Zeno's paradoxes: if the race continues the same way,

$$50 \text{ meters} + 25 \text{ meters} + \dots$$

will it be true that the runners will never reach the the end of the 4-th circle (one circle is 400 meters)? (My father was retelling my question to his fellow officers; before World War II, he graduated from the Mathematics Department of Saratov University).

Little Sasha was walking on the edge of an abyss; being a trained mathematician, his father had a false sense of security because perhaps he believed that Zeno's "arrow" paradox (of which Swedish relay is an obvious version) is resolved in elementary calculus by summation of the geometric progression

$$\begin{aligned} 800 + 400 + 200 + 100 + 50 + 25 + \dots &= 1600 \\ &= 4 \times 400. \end{aligned}$$

This is true; the runners will indeed reach the end of the 4th circle, and fairly quickly.

But if you think that Zeno's paradox ends here, you are wrong; be prepared to face one of its most vicious forms. Indeed, the real trouble starts after the successful finish of the race: *where is the baton?* Indeed, the whole point of the relay is that each runner passes the baton to the runner on the next leg. After the race is over, each runner can honestly claim that he is no longer in possession of the baton because he passed it to the next runner.

I repeat: can you explain *where is the baton?*

A spoiler is in Section 11 at the end of the paper.

The baton paradox is a version of a *supertask* invented by Jon Pérez Laraudogoitia [12] and is of serious importance for discussion of foundations of statistical mechanics. I borrow its compact description from Zurab Silagadze's survey [16] of Zeno type paradoxes as they appear in modern science:

In [12] Pérez Laraudogoitia constructed a beautifully simple supertask which demonstrates some weird things even in the context of classical

mechanics. Imagine an infinite set of identical particles arranged in a straight line. The distance between the particles and their sizes decrease so that the whole system occupies an interval of unit length. Some other particle of the same mass approaches the system from the right with unit velocity. In elastic collision with identical particles the velocities are exchanged after the collision. Therefore a wave of elastic collisions goes through the system in unit time. And what then? Any particle of the system and the projectile particle comes to rest after colliding its left closest neighbor. Therefore all particles are at rest after the collision supertask is over and we are left with paradoxical conclusion [12] that the total initial energy (and momentum) of the system of particles can disappear by means of an infinitely denumerable number of elastic collisions, in each one of which the energy (and momentum) is conserved!



FIGURE 5. *Energy and momentum are disappearing in Pérez Laraudogoitia's infinite sequence of collisions.*

In adult terms, the baton paradox is one instant of the great and difficult controversy of potential infinity *vs* actual infinity; but, as we see, the problem can be formulated in terms accessible to a child. I think that in the ideal world, a teacher of mathematics should follow the dictum from J. D. Salinger's *Catcher in the Rye* and gently guide children through mathematics guarding them from dangers but not concealing their existence:

I keep picturing all these little kids playing some game in this big field of rye and all. Thousands of little kids, and nobody's around—nobody big, I mean—except me. And I'm standing on the edge of some crazy cliff. What I have to do, I have to catch everybody if they start to go over the cliff—I mean if they're running and they don't look where they're going. I have to come out from somewhere and catch them. That's all I'd do all day. I'd just be the catcher in the rye and all.

10. *Conclusions*

And returning to spirituality of mathematics, I wish to make my final comment:

We should not underestimate the intellectual courage of children; we

should not underestimate the power of a childish view of the world—and we should not underestimate the difficulty of mathematics of the infinite, either.



FIGURE 6. *A study for Vision to the Youth Bartholomew by Mikhail Vasilyevich Nesterov. We should not underestimate the intellectual courage of children, and we should not underestimate the power of a childish view of the world. Public domain.*

I am not prepared to accept the wisdom of the words:

When I was a child, I spake as a child, I understood as a child, I thought as a child: but when I became a man, I put away childish things. (1 Corinthians 13:11)

Indeed I stand for my former (or inner?) child. In my stance, I find support in words of Michael Gromov (for those not in the know: he is a *really* famous and great mathematician):

My personal evaluation of myself is that as a child till 8–9, I was intellectually better off than at 14. At 14–15 I became interested in math.

It took me about 20 years to regain my 7 year old child perceptiveness.

Acknowledgements

The list of people who helped me in many ways in my work on the project reflected in this paper is becoming almost as long as the paper itself; I refer the reader to *Introduction* of my book *Shadows of the Truth* [5].

Disclaimer

The author writes in his personal capacity and the views expressed do not necessarily represent position of his employer or any other person, organisation or institution.

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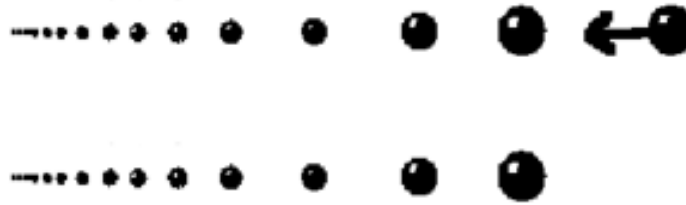


FIGURE 7. *Position of the balls before and after the collisions.*

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11. *The spoilers*

The solution to the baton paradox is simple: the baton has slipped from runners' hands and continues to fly beyond the finishing line. To come to this conclusion, we have to assume that the laws of conservation of momentum and energy survive the transition from finite to (imaginary) infinite systems of physical objects and work at arbitrary small time and space scales. Of course, mathematics can be applied to imaginary worlds, but only if they are consistent. But is the world of an infinite the Swedish relay race consistent?

So, are these assumptions acceptable to you? Are you convinced?

It is interesting to compare the baton paradox with Laraudogoitia's paradox. Let us look at the positions of balls before and after collisions (when all balls are motionless), Figure 7.

The total mass of the balls is infinite. What we see is a ball striking a very massive (actually, infinitely massive) wall. The wall does not move, but absorbs the energy and momentum of the impact.

This explanation requires, however, (imaginary) extension of Newton's Laws to (imaginary) mechanical systems which include objects of infinite mass. But is this extension consistent?

Again, is this assumption acceptable to you? Are you convinced?