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Shadows of the Truth: Metamathematics of Elementary Mathematics

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To Noah and Emily

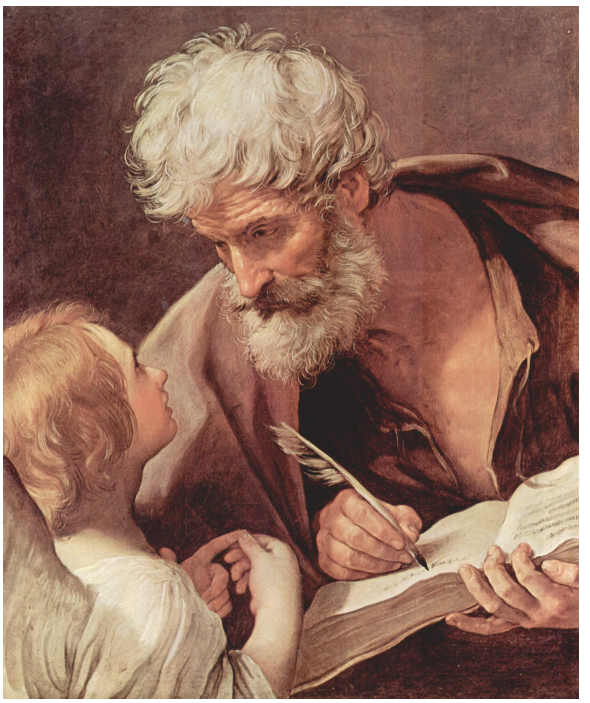

Fig. 0.1. *L'Evangelista Matteo e l'Angelo.* Guido Reni, 1630–1640. Pinacoteca Vaticana. Source: *Wikipedia Commons*. Public domain.

Guido Reni was one of the first artists in history of visual arts who paid attention to psychology of children. Notice how the little angel counts on his fingers the points he is sent to communicate to St. Matthew.

Preface

Toutes les grandes personnes ont d'abord été des enfants
(Mais peu d'entre elles s'en souviennent.)
Antoine de Saint-Exupéry, *Le Petit Prince.*

This book is an attempt to look at mathematics from a new and somewhat unusual point of view. I have started to systematically record and analyze from a *mathematical* point of view various difficulties experiencing by children in their early learning of mathematics. I hope that my approach will eventually allow me to gain a better understanding of how we—not only children, but adults, too—do mathematics. This explains the title of the book: *metamathematics* is mathematics applied to study of mathematics. I chase shadows: I am trying to identify and clearly describe hidden structures of elementary mathematics which may intrigue, puzzle, and—like shadows in the night—sometimes scare an inquisitive child.

The real life material in my research is limited to stories that my fellow mathematicians have chosen to tell me; they represent tiny but personally significant episodes from their childhood. I directed my inquiries to mathematicians for an obvious reason: only mathematicians possess an adequate language which allows them to describe in some depths their experiences of learning mathematics. So far my approach is justified by the warm welcome it found among my mathematician friends, and I am most grateful to them for their support. For some reason (and the reason deserves a study on its own) my colleagues know what I am talking about!

The book was born from a chance conversation with my colleague Elizabeth Kimber. I analyze her story, in great detail, in Chapter 5. Little Lizzie, aged 6, could easily solve "put a number in the box" problems of the type

$$7 + \square = 12,$$

by counting how many 1's she had to add to 7 in order to get 12 but struggled with

$$\square + 6 = 11,$$

because she did not know where to start. Much worse, little Lizzie was frustrated by the attitude of adults around her—they could not comprehend her difficulty, which remained with her for the rest of her life.

When I heard that story, I instantly realized that I had had similar experiences myself, and that I heard stories of challenge and frustration from many my fellow mathematicians. I started to ask around—and now offer to the reader a selection of responses arranged around several mathematical themes.

A few caveats are due. The stories told in the book cannot be independently corroborated or authenticated—they are memories that my colleagues have chosen to remember. I believe that the stories are of serious interest for the deeper understanding of the internal and hidden mechanisms of mathematical practice because the memories told have deeply personal meaning for mathematicians who told the stories to me. The nature of this deep emotional bond between a mathematician and his or her first mathematical experiences remains a mystery—I simply take the existence of such a bond for granted and suggest that it be used as a key to the most intimate layer of mathematical thinking.

This bond with the "former child" (or the "inner child"?) is best described by Michael Gromov:

> I have a few recollections, but they are not structural.
>
> I remember my feeling of excitement upon hitting on some mathematical ideas such as a straight line tangent to a curve and representing infinite velocity (I was about 5, watching freely moving thrown objects). Also at this age I was fascinated by the complexity of the inside of a car with the hood lifted.
>
> Later I had a similar feeling by imagining first infinite ordinals (I was about 9 trying to figure out if 1000 elephants are stronger than 100 whales and how to be stronger than all of them in the universe).
>
> Also I recall many instances of acute feeling of frustration at my stupidity of being unable to solve very simple problems at school later on.
>
> My personal evaluation of myself is that as a child till 8–9, I was intellectually better off than at 14. At 14–15 I became interested in math. It took me about 20 years to regain my 7 year old child perceptiveness.

I repeat Michael Gromov's words:

> *It took me about 20 years to regain my 7 year old child perceptiveness.*

I am confident that this sentiment is shared by many my mathematician colleagues. This is why I concentrate on the childhood of mathematicians, and this is why I expect that my notes will be useful to specialists in mathematical education and in psychology of education. But I wish to make it absolutely clear: I am *not* making any recommendations on mathematics teaching. Moreover, I emphasize that the primary aim of my project is to understand the nature of mainstream "research" mathematics.

The emphasis on children's experiences makes my programme akin to linguistic and cognitive science. However, when a linguist studies formation of speech in a child, he studies language, not the structure of linguistics as a scientific discipline. When I propose to study the formation of mathematical concepts in a child, I wish to get insights into the interplay of mathematical structures in *mathematics*. Mathematics has an astonishing power of reflection, and a self-referential study of mathematics by mathematical means plays an increasingly important role within mathematical culture. I simply suggest to take a step further (or a step aside, or a step back in life) and to take a look back in time, at one's childhood years.

A philosophically inclined reader will immediately see a parallel with Plato's Allegory of the Cave: children in my book see shadows of the Truth and sometimes find themselves in a psychological trap because their teachers and other adults around them see neither Truth, nor its shadows. But I am not doing philosophy; I am a mathematician and I stick to a concise mathematical reconstruction of what the child had actually seen.

My book is also an attempt to trigger the chain of memories in my readers: even the most minute recollection of difficulties and paradoxes of their early mathematical experiences is most welcome. Please write to me at

borovik@manchester.ac.uk.

BIBLIOGRAPHY. At the end of each chapter I place some bibliographic references. Here are some (very different) books most closely related to themes touched on in this introduction: Aharoni [610], Carruthers and Worthington [642, 644], Freudenthal [667], Gromov [30], Krutetskii [827], and McManaman and Droujkova [720].

Alexandre Borovik
Didsbury
12 February 2014

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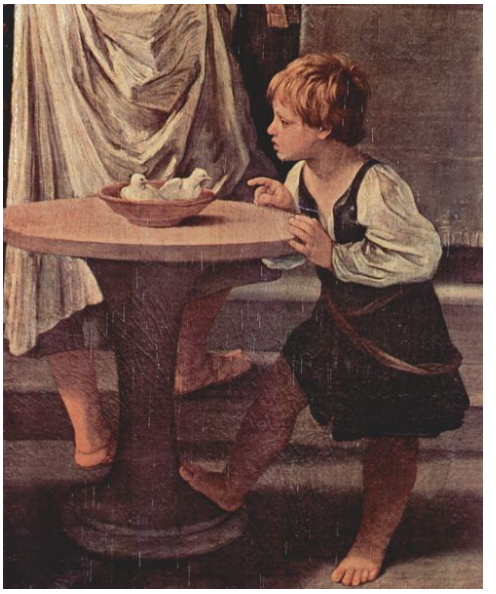

Fig. 0.2. Guido Reni. A fragment of *Purification of the Virgin*, c. 1635–1640. Musée du Louvre. Source: *Wikipedia Commons*. Public domain.

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"philanthropic catalyst for discovery in areas engaging in life's biggest questions."

However, the opinions expressed in the book are those of the author and do not necessarily reflect the views of the John Templeton Foundation.

Finally, my thanks go to the blogging community—I have picked in the blogosphere some ideas and quite a number of references—especially to the late Dima Fon-Der-Flaass and to my old friend who prefers to be known only as Owl.

Alexandre Borovik
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